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Network Flow Circulation with Risk Function

Vassil Sgurev, Lyubka Doukovska, Stanislav Drangajov

Institute of Information and Communication Technologies – BAS, Sofia, Bulgaria
Emails: vsgurev@gmail.com; l.doukovska@mail.bg; sdrangajov@gmail.com

Abstract: The present paper proposes the introduction of a risk function in the circulation in a network flow. Risk is defined as the product of the arc flow and the probability of an adverse event in that arc flow. It has been proven that in this case the network flow is two-product and consists of resources and risks with separate capacities and arc estimates for each of them. A method has been developed for determining the circulation with a minimum value on a two-product network flow with a fixed value, as well as the maximum possible circulation with a minimum value. Relevant theoretical results for the behaviour of the two-product network flow have been obtained. A numerical example is given, which illustrates the obtained research results. The example shows the possibility for practical use of the network-flow two-product circulation in closed logistics systems.

Keywords: *Two-product network flow, risk function, generalized network flow, maximum circulation with minimum value, closed logistics system*.

1. Introduction

In recent years, interest in the emergence and assessment of risk in various areas of material production and public practice has increased [6]. To solve these problems, it is necessary to be aware with a number of theoretical and applied aspects of risk. There are differences in the definition of risk in technogenious and natural disasters and catastrophes, as well as in the financial and economic sphere.

In [4, 5, 7] the use of the risk function in the network-flow models is proposed. The method for risk assessment in technogenious and natural disasters and catastrophes – earthquakes, floods, fires, road accidents, etc. has been adopted. In this case, the risk is defined as the product of two measures – the volume of the resource (flow) on a given section (arc) of the network and the probability of an adverse event in that section. It follows that risk is considered as a resource (flow) and is measured in the same units. Then on the same network can be realized a flow of two products – resources and risks. The resource, as a product, must meet the requirements for "flowability", i.e. for equality of the volumes of flow that enters in and leaves from a given vertex (node) of the network. For the product "risk", the requirement for flowability may not be observed.

In many applications, for example in closed logistics systems, the circulation of vehicles between different vertices for freight, transport and logistics activities is required, taking into account the emerging risks. In this case, it is expedient to use as a model the circulation of a two-product network flow with a risk function and with capacities on the individual arcs (sections). This makes it possible to solve various optimization problems – for the maximum possible circulation, for the maximum possible circulation of minimum cost, for circulation of minimum value with any (below the maximum feasible) value of the flow, etc. On the other hand, the cost of the total risk can be either fixed or or variable. Such an approach makes it possible to minimize the total amount of costs for transporting resources and for risk insurance at flow circulation.

If discrete transport units are used for the transportation of resources, it is necessary to take into account the time of passage along the individual sections (arcs) of the network in the network-flow model for circulation, i.e. to conditionally switch from a discrete to a continuous network-flow model. In this case, the use of a generalized two-product network flow with capacities and risk function is proposed.

2. Formalized Model and Methodology

For the formal description of the network-flow models for circulation with a risk function, it is necessary to introduce appropriate denotations [1, 2, 3] as follows:

- *G(X, U)* – directed graph with a set of vertices (nodes) $X = \{x_i, i \in I\}$ and a set of arcs (edges) $U = \{x_{ij}, (i,j) \in G\}$, where I is the set of indices of all vertices $X - I = \{ i, x_i \in X\}$, and G is the set of pairs of indices of all arcs U of the graph – $G = \{(i,j),\ x_{i,j} \in U\}$. Each arc $x_{i,j}$ has an initial vertex x_i and end vertex x_j;
- f_{ij} – flow function of the resource on arc x_{ij};
- r_{ij} – flow function of risk on the arc x_{ij}, or for short risk on this arc;

$$r_{ij} = p_{ij}\, f_{ij},\ 0 \le p_{ij} \le 1, (i,j) \in G \tag{1}$$

- p_{ij} – probability an adverse event to evolve at transportation of resource f_{ij} along arc x_{ij}.

Value risk, r_{ij} is defined as a product of two measures – the amount of resource which passes along the arc and the probability for an adverse evet at that transition.

The following denotation are adopted concerning the directed graph $G(X, U)$: $\Gamma_i^1 = \{j/(i,j)\} \in G$ – direct mapping, consisting of the indices of end vertices of all arcs outgoing from vertex x_i;

- $\Gamma_i^{-1} = \{j/(i,j)\} \in G$ – reverse mapping, representing the set of indices of all initial vertices of arcs whose end vertex is x_i;
- c'_{ij} – upper bound of capacity for resource on arc x_{ij};
- c''_{ij} – upper bound of capacity for risk on arc x_{ij};
- a'_{ij} – cost for passing a unit of resource along arc x_{ij};
- a''_{ij} – cost for passing a unit of risk along arc x_{ij};
- v and v_{max} – flow and maximal flow of resources;
- v' and v'_{max} – flow and maximal flow of risks.

To observe constraints for resources $\{c'_{ij}\}$ and risks $\{c''_{ij}\}$ it is necessary to introduce [4, 7]

$$c^s_{ij} = \min[c'_{ij}, \frac{c''_{ij}}{p_{ij}}],\ (i,j) \in G \qquad (2)$$

c^s_{ij} – coordinated capacity for resources and when observed both capacity for resources – c'_{ij} and risks – c''_{ij} are observed;

$$c^r_{ij} = \min[p_{ij}c'_{ij}, c''_{ij}],\ (i,j) \in G \qquad (3)$$

c^r_{ij} – coordinated capacity for risks and when observed both capacity for risks – c''_{ij} and for resources – c'_{ij} and are observed.

If both sides of the first of the two equalities above are multiplied by the probability p_{ij}, then the following two equalities will be obtained [7]

$$c^s_{ij}p_{ij} = \min\left[p_{ij}c'_{ij}, c''_{ij}\right] = c^r_{ij} \qquad (4)$$

or

$$c^r_{ij} = c^s_{ij}p_{ij};\ c^s_{ij} = \frac{c^r_{ij}}{p_{ij}} \qquad (5)$$

If both network products – resource $\{f_{ij}\}$ and risk $\{r_{ij}\}$ are upper bounded by the inequalities:

$$f_{ij} \le c^s_{ij};\ r_{ij} \le c^r_{ij};\ \ (i\ ,j) \in G \qquad (6)$$

then it follows from the previous four equalities that the next inequalities will be also observed: for each $(i,j) \in G$

$$f_{ij}\ \le c'_{ij},\ \ r_{ij}\ \le c''_{ij}, \qquad (7)$$

The following denotations are needed for the realization of flow circulation:

- $I_s = \{i/x_i \in S\}$ – indices of the set of loading points;

$$S = \cup\ x_i,\ i\ \in\ I_s \qquad (8)$$

- a_i – capacity (performance) by resources of a loading point of index $i \in I_s$;
- $I_t = \{i/x_i \in T\}$ – indices in the set of loading points;

$$T = \cup x_i,\ i \in I_t \tag{9}$$

- b_i – capacity (performance) by resources of *a* unloading point of index $i \in I_t$;
- $I_p = \{i/x_i \in P\}$ – indices of the set of intermediate points;

$$P = \cup x_i,\ i \in I_p \tag{10}$$

The following requirement is to be observed for the product being transported:

$$\sum_{i \in I_s} a_i = \sum_{j \in I_t} b_j \tag{11}$$

Let l_{ij} be the number of time units for traveling the distance from point x_i to point x_j, which includes the time for loading/unloading at the same point. Then k_{ij} it is the coefficient for transition from discrete to continuous flow, i.e.

$$k_{ij} = \frac{1}{l_{ij}}, (i,j) \in G. \tag{12}$$

If we come from the insurance costs, it is expedient to assume that the risk r_{ij} should not be less than the real arc risk $p_{ij}f_{ij}$, obtained through the resource flow f_{ij}, i.e. it is necessary to observe inequalities [4, 5, 6]

$$p_{ij}f_{ij} \leq r_{ij}; (i,j) \in G \tag{13}$$

which leads to

$$p_{ij}f_{ij} - r_{ij} \leq 0; (i,j) \in G. \tag{14}$$

The requirement above leads to the possibility to build a network risk flow, although with some increase in the risk on individual arcs. This allows the definition of a two-product network flow of resources and risks with a total minimum cost.

The total costs for resource and risks are equal to:

$$L_s = \sum_{(i,j) \in G} a'_{ij} f_{ij} \tag{15}$$

$$L_r = \sum_{(i,j) \in G} a''_{ij} r_{ij} \tag{16}$$

$$L = L_s + L_r = \sum_{(i,j) \in G} a'_{ij} f_{ij} + \sum_{(i,j) \in G} a''_{ij} r_{ij} \rightarrow min\,|max \tag{17}$$

where L_s are expenses for transportation of resources, L_r – expenses for insurance of risks, L – total expenses $(L_s + L_r)$.

The value L can be considered as an objective function in the problems for optimization of the total costs of the two-product flow.

If the transport of the resource is carried out with discrete transport units and the travel times along the different arcs of the set U are different, then the circulation should be carried out using a generalized two-product network flow with risks, capacities and coefficients $\{k_{ij}/(i,j) \in U\}$.

The denotations introduced allow for the formalization of the circulation of the two-product network flow of resources and risks.

Problem A: Determining the minimum value of the circulation in the two-product flow with fixed values of resources and risks in the generalized two-product flow with fixed values of $\sum_{i \in I_s} a_i$ will be reduced to the following linear programming problem [7]:

$$L = \sum_{(i,j)\in G} a'_{ij} f_{ij} + \sum_{(i,j)\in G} a''_{ij} r_{ij} \to min \tag{18}$$

Subject to constraints: for each $i \in I$ and $(i,j) \in G$

$$\sum_{j\in\Gamma_i^1} f_{ij} - \sum_{j\in\Gamma_i^{-1}} k_{ji} f_{ji} = 0, \qquad i \in I \tag{19}$$

$$\sum_{j\in\Gamma_i^{-1}} f_{ji} = a_i, i \in I_s \tag{20}$$

$$\sum_{j\in\Gamma_i^{-1}} f_{ji} = b_i, i \in I_t \tag{21}$$

$$p_{ij} f_{ij} - r_{ij} \le 0;\ (i,j) \in G \tag{22}$$

$$f_{ij} \ge 0;\ (i,j) \in G \tag{23}$$

$$\sum_{j\in\Gamma_i^1} r_{ij} - \sum_{j\in\Gamma_i^{-1}} r_{ji} = 0, i \in I \tag{24}$$

$$r_{ij} \le c_{ij}^r; (i,j) \in G \tag{25}$$

$$r_{ij} \ge 0;\ (i,j) \in G \tag{26}$$

The value

$$\sum_{(i,j)\in G} f_{ij}^{*} = f^{*} \tag{27}$$

shows the minimum number of transport units that can ensure through circulate the transport of the resource $\sum_{i\in I_s} a_i$ from the loading points S to the unloading points T, with minimal transportation and risk insurance costs.

Problem B: Determining of the maximum possible circulation of the generalized two-product network flow with risks can be performed by the following linear programming problem:

$$L = v_{max} = \left(\sum_{(i,j)\in G} f_{ij} + \sum_{(i,j)\in G} r_{ij}\right) \rightarrow max \tag{28}$$

observing constraints (19) and from (22) to (26).

The obtained value v_{max} for the maximum possible circulation allows to use a number of dependences on the classical network flow, including the *mincut-maxflow* theorem of Ford-Fulkerson [1], namely:

$$v_{max} = f(X_0, \overline{X_0}) = c(X_0, \overline{X_0}) \tag{29}$$

$$f(\overline{X_0}, X_0) = \emptyset \tag{30}$$

$$(X_0, \overline{X_0}) = \{x_{ij}/x_i \in X_0;\ x_j \in \overline{X_0}\} \tag{31}$$

$$(\overline{X_0}, X_0) = \{x_{ij}/x_i \in \overline{X_0};\ x_j \in X_0\} \tag{32}$$

where $(X_0, \overline{X_0})$ is the minimal cut.

Problem C: If the maximum flow v_{max} in circulation is determined by **Problem B**, then the maximum flow with a minimum value can be obtained through the following linear programming problem, where the objective function is (18), the constraints – (19), from (22) to (22) to (26) and one additional equality

$$\sum_{(i,j)\in G} f_{ij} + \sum_{(i,j)\in G} r_{ij} = v_{max} \tag{33}$$

where v_{max} is the constant value from **Problem B.**

3. Numerical Example

A graph $G(X, U)$ with four vertices and five arcs is shown in Fig. 1.

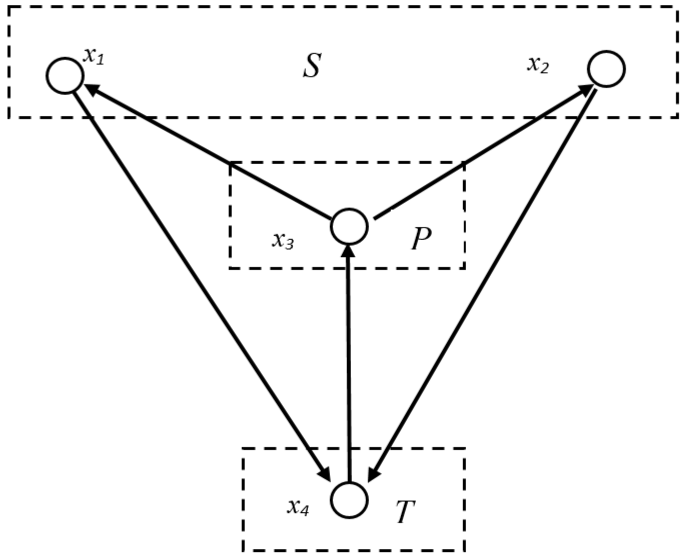

Fig. 1. Exemplary graph

$$X = \{x_1, x_{21}, x_3, x_4\} \tag{34}$$

$$U = \{x_{31}, x_{32}, x_{14}, x_{24}, x_{43}\} \tag{35}$$

Equation (11) takes the following form:

$$\sum_{i \in I_s} a_i = \sum_{j \in I_t} b_i = 3$$

A total value of the flow of 3 units is assumed for **Problem A**. The source data are given in Table 1.

Table 1. Source data

Para-meters	Arcs – resource					Arcs – risks				
	(3,1)	(3,2)	(1,4)	(2,4)	(4,3)	(3,1)	(3,2)	(1,4)	(2,4)	(4,3)
c'_{ij}	10	8	5	5	4	–	–	–	–	–
c''_{ij}	–	–	–	–	–	2.2	1.2	1.4	0.8	1.8
c^s_{ij}	10	4	5	2.67	4	–	–	–	–	–
c^r_{ij}	–	–	–	–	–	2	1.2	0.5	0.8	1.4
a'_{ij}	10	9	11	13	14	–	–	–	–	–
a''_{ij}	–	–	–	–	–	12	8	7	5	8
p_{ij}	0.2	0.3	0.1	0.3	0.25	0.2	0.3	0.1	0.3	0.25

Equations (18) to (26) take the following form:

Objective function:

$$\min L = 10f_{31} + 9f_{32} + 11f_{14} + 13f_{24} + 14f_{43} + \\ + 12r_{31} + 8r_{32} + 7r_{14} + 5r_{24} + 8r_{43} \quad (34)$$

subject to constraints:

$$f_{14} - f_{31} = 0 \quad (35)$$

$$f_{24} - f_{32} = 0 \quad (36)$$

$$f_{31} + f_{32} - f_{43} = 0 \quad (37)$$

$$f_{43} - f_{14} - f_{24} = 0 \quad (38)$$

$$f_{31} + f_{32} = 3 \quad (39)$$

$$f_{14} + f_{24} = 3 \quad (40)$$

$$0.2\, f_{31} - r_{31} <= 0 \quad (41)$$

$$0.3\, f_{32} - r_{32} <= 0 \quad (42)$$

$$0.1\, f_{14} - r_{14} <= 0 \quad (43)$$

$$0.3\, f_{24} - r_{24} <= 0 \quad (44)$$

$$0.25\, f_{43} - r_{43} <= 0 \quad (45)$$

$$r_{14} - r_{31} = 0 \quad (46)$$

$$r_{24} - r_{32} = 0 \quad (47)$$

$$r_{31} + r_{32} - r_{43} = 0 \quad (48)$$

$$r_{43} - r_{14} - r_{24} = 0 \quad (49)$$

$$r_{31} <= 2. \quad (50)$$

$$r_{32} <= 1.2 \quad (51)$$

$$r_{14} <= 0.5 \quad (52)$$

$$r_{24} <= 0.8 \quad (53)$$

$$r_{43} <= 1 \quad (54)$$

$$f_{ij} \geq 0;\ (\mathrm{i,j}) \in \mathrm{G} \quad (55)$$

$$r_{ij} \geq 0;\ (\mathrm{i,j}) \in \mathrm{G} \quad (56)$$

The calculated value of the objective function is 124.05.

The results of the solutions of **Problem A** from the numerical example of Fig. 1 are shown in Table 2.

Table 2. Results of Problem A

Para-meters	Arcs – resource					Arcs – risks				
	(3,1)	(3,2)	(1,4)	(2,4)	(4,3)	(3,1)	(3,2)	(1,4)	(2,4)	(4,3)
f_{ij}	1.5	1.5	1.5	1.5	3	–	–	–	–	–
r_{ij}	–	–	–	–	–	0.3	0.45	0.3	0.45	0.75
$c'_{ij} - f_{ij}$	8.5	2.5	3.5	2.5	1	–	–	–	–	–
$c''_{ij} - r_{ij}$	–	–	–	–	–	1.9	0.75	1.10	0.35	1.05
p_{ij}	0.2	0.3	0.1	0.3	0.25	0.2	0.3	0.1	0.3	0.25
$c^s_{ij} - f_{ij}$	8.5	2.5	3.5	1.17	1	–	–	–	–	–
$c^r_{ij} - r_{ij}$	–	–	–	–	–	1.7	0.75	1.10	0.75	1.5

As can be seen from Table 2, there is no saturated arc in either resources or risk, which means that the total flow can be increased. There is also no minimum cut.

In the next **Problem B**, as indicated in the previous section, the maximum total flow is sought and the objective function is:

$$L = v \rightarrow max \quad (56)$$

by introducing the following constraints

$$v = v_1 + v_2 \quad (57)$$

$$f_{14} = v_1 \quad (58)$$

$$f_{24} = v_2 \quad (59)$$

and constraints (39) and (40) of the previous **Problem A** are replaced by

$$f_{31} + f_{32} = v \quad (60)$$

$$f_{14} + f_{24} = v \quad (61)$$

at which the maximum possible flow $v = 4$ is achieved.

Table 3 shows the existence of saturated arcs for the circulation both by resource – f_{43} and by risk.

In the last **Problem C** for maximum flow of minimum cost the objective function, as well as the constraints are identical to those in **Problem A** with the only change that in constraints (39) and (40), the value of the flow is already 4 – the maximum possible, and the value of the objective function it is already 165.4.

Table 3. Results maxflow-mincost

Para-meters	Arcs – resource					Arcs – risks				
	(3,1)	(3,2)	(1,4)	(2,4)	(4,3)	(3,1)	(3,2)	(1,4)	(2,4)	(4,3)
f_{ij}	2.5	1.5	2.5	2.5	4	–	–	–	–	–
r_{ij}	–	–	–	–	–	0.5	0. 5	0.5	0. 5	1
$c'_{ij} - f_{ij}$	7.5	6.5	3.5	2.5	1	–	–	–	–	–
$c''_{ij} - r_{ij}$	–	–	–	–	–	1.5	0.7	09	0.3	0.8
p_{ij}	0.2	0.3	0.1	0.3	0.25	0.2	0.3	0.1	0.3	0.25
$c^s_{ij} - f_{ij}$	7.5	2.5	3.5	1.17	**0**	–	–	–	–	–
$c^r_{ij} - r_{ij}$	–	–	–	–	–	1.5	0.7	**0**	0.3	0.4

As can be seen from the arc flow values, the flow is slightly changed and in order to obtain a minimum pass cost, the values of the flows f_{14} and f_{24} have been changed.

4. Conclusion

1. It is proposed to introduce into the circulation of a network flow a risk function, which takes into account the probabilities of adverse events on different sections of the network.
2. It has been shown that this leads to a two-product generalized network flow of resources and risks with separate capacities for both products.
3. A method is proposed for determining the circulation with a total minimum value for both products of the network flow – for resources and risks. A method for determining the maximum possible circulation in the two-product network flow is described, as well as the maximum circulation with a minimum cost in the same flow.
4. The obtained theoretical results are confirmed by an appropriate numerical example. It is shown how these results can be used in closed logistics systems.

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References

1. Ford, L. R., Fulkerson, D. R.: Maximal flow through a network. Canadian Journal of Mathematics 8, 399–404 (1956).

2. Jensen, P. A., J. W. Barnes: Network flow programming. John Wiley and Sons, Inc., New York (1980).
3. Christofides, N.: Graph theory: An Algorithmic Approach. Academic Press, (1975).
4. Sgurev, V., St. Drangajov.: Intelligent control of flows with risks on a network. In: Filev D. et al. (eds) Intelligent Systems'2014. Advances in Intelligent Systems and Computing, vol 323, Springer, Cham. https://doi.org/10.1007/978-3-319-11310-4_3 (2015).
5. Sgurev, V., S. Drangajov.: Optimal control of the flow of risk on network. In: Proc. of International Conference "Automatics and Informatics'15", pp. 119–122, Bulgaria, (2015).
6. Beck, U.: World at Risk. English Edition, Polity Press, (2009).
7. Sgurev, V., S. Drangajov.: Network Flows with Risks. Prof. Marin Drinov Publishing House of BAS, Sofia, (in Bulgarian) (2019).