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A Reference Neighbourhood Interactive Algorithm for Solving Multicriteria Linear Integer Programming Problems

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1. Introduction

The interactive algorithm [2] are widely used for solving multicriteria linear integer programming problems. Some of the interactive algorithms work with the aspiration levels of the criteria, others use weights to denote the relative significance of the criteria. We propose a learning-oriented [4] interactive algorithm for solving multicriteria linear integer programming problems. The main features of the algorithm proposed, which to a large extent preserves the positive aspects of the available interactive algorithms [2] and improves the dialogue with the DM, are:

–at every iteration the DM provides his/her local preferences in terms of the desired changes in the criteria values of some of the criteria, the desired directions of change of the other criteria and the values and/or directions of the eventual deterioration of the remaining criteria, instead of aspiration levels of the criteria. The current preferred solution and the local preferences of the DM define a reference neighborhood in which the next preferred solution is searched for;

–at every iteration in a reference neighbourhood a set of integer near (weak) nondomnated solutions or integer (weak) nondominated solution is searched for solving integer scalarizing problems.

2. Problem formulation

The multicriteria linear integer programming (I) can be formulated as

(1) $$\text{"max"}\{f_k(x),\ k \in K\}$$

subject to:

(2) $$\sum_{j\in N} a_{ij}x_j \ \mathrm{I}\ b_i,\ i \in M,$$

(3) $0 \leq x_j \leq d_j,\ j \in N,$

(4) x_j - integer, $j \in N,$

where symbol "max" means that all the objective functions are to be simultaneously maximized; $K = \{1, 2, ..., p\}$, $M = (1, 2, ..., m\}$, $N = \{1, 2, ..., n\}$ denote the index sets of the objective functions (criteria), the linear constraints, and the decision variables, respectively: $f_k(x)$, $k \in K$ are linear criteria (objective functions); $f_k(x) = \sum_{j \in N} c_j^k x_j$ and $x = (x_1, x_2, ..., x_j, ..., x_n)^T$ is the vector of the decision variables.

The constraints (2)–(4) define the feasible region X_1 for the integer variables.

For clarity of exposition, we introduce a few definitions:

Definition 1. The solution x is called efficient solution of problem (I) or (P), if there does not exist any other solution x' so that the following inequalities are satisfied:

$f_k(x') \geq f_k(x)$ for every $k \in K$ and

$f_k(x') > f_k(x)$ at least for one index $k \in K$.

Definition 2. The solution x is called a weak efficient solution of problem (1) or (P) if there does not exist another solution x' such that the following inequalities are fulfilled:

$f_k(x') > f_k(x)$ for every $k \in K$.

Definition 3. The solution x is called a (weak) efficient solution, if x is either an efficient or a weak efficient solution.

Definition 4. The vector $f(x) = (f_1(x), ..., f_p(x))^T$ is called a (weak) nondominated solution in the criteria space, if x is a (weak) efficient solution in the variables space.

Definition 5. A near (weak) nondominated solution is a feasible solution in the criteria space located comparatively close to the (weak) nondominated solutions.

Definition 6. A current preferred solution is a near (weak) nondominated solution or (weak) nondominated solution chosen by the DM at the current iteration. The most preferred solution is a preferred solution that satisfies the DM to the greatest degree.

Definition 7. Desired changes of the criteria values are the amounts by which the DM wishes to increase the criteria in comparison with their value in the current preferred solution.

Definition 8. The desired directions of change of the criteria are the directions, in which the DM wishes to improve the criteria in comparison with their values at the current preferred solution.

Definition 9. Reference neighbourhood is defined by the current preferred solution; the desired changes in the values of some of the criteria, the desired directions of change of the other criteria and the values and/or directions of the eventual deterioration of the remaining criteria as specified by the DM.

Problems (I) do not possess a mathematically well-defined optimal solution. Hence it is necessary to select one of the (weak) nondominated solutions, which is most appropriate for the global DM's preferences. This choice is subjective and depends entirely on the DM.

3. Scalarizing problems

We formulate the scalarazing problems [1, 8] under the assumption that the set of criteria K can be divided into three subsets – K_1, K_2 and K_3. The set K_1 contains the indices $k \in K$ of the criteria for which the DM wants to improve their values compared to the values in the current preferred solution. The set K_2 includes the indices $k \in K$ of the criteria for which the DM agrees to worsen their values not setting the exact values of deterioration. The set K_3 contains the indices $k \in K$ of the criteria whose values the DM wants to preserve or agrees to be worsened by the value δ_k. The set K_1 is divided into two subsets – K'_1 and K''_1; K'_1 contains indices of the criteria $k \in K_1$ that the DM wants to improve by desired values, Δ_k and K''_1 consists of indices of the criteria $k \in K_1$, that the DM wants to improve and for which he/she is not able to set the exact values of improving.

The following scalarizing problem, named E_1, is proposed to obtain a (weak) nondominated solution of the multicriteria integer problem (I) in the reference neighbourhood of the current preferred solution.

Minimize

(5) $$S(x) = \max[\max_{k \in K'_1}(\bar{f}_k - f_k(x))/|f'_k|,\ \max_{k \in K_2}(f_k - f_k(x))/|f'_k|] + \max_{k \in K''_1}(f_k - f_k(x))/|f'_k|,$$

subject to

(6) $$f_k(x) \geq \tilde{f}_k,\ k \in K_3 \cup K''_1,$$

(7) $$x \in X_1,$$

where

f_k – the value of the criterion with an index $k \in K$ in the current preferred solution, $\bar{f}_k = f_k + \Delta_k$ is the desired level of the criterion with an index $k \in K'_1$;

$$\tilde{f}_k = \begin{cases} f_k, \text{ if } k \in K''_1, \\ f_k, \text{ if } k \in K_3 \text{ and the DM wants to preserve the current value of the criteria} \\ \qquad \text{with index } k, \\ f_k - \delta_k, \text{ if } k \in K_3 \text{ and the DM agrees to worsen with value } \delta_k \text{ the current} \\ \qquad \text{value of the criteria with an index } k, \end{cases}$$

f'_k – a scaling coefficient,

$$f'_k = \begin{cases} f_k, \text{ if } f_k \neq 0, \\ 1, \text{ if } f_k = 0. \end{cases}$$

Theorem 1. The optimal solution of the scalarizing problem E_1 is a weak efficient solution of the multicriteria integer programming problem (*I*).

Proof.

Let K'_1 and $K''_1 \neq \varnothing$.

Let x^* be an optimal solution of problem E_1. Then the following condition is satisfied:

$$S(x^*) \leq S(x),\ x \in X,$$

and $f_k(x^*) \geq \tilde{f}_k$, $k \in K_1'' \cup K_3$.

Let us assume that x^* is not a weak Pareto optimal solution of the initial multiple criteria integer problem (I). In this case there must exist, $x' \in X$ for which:

(8) $$f_k(x') > f_k(x^*) \text{ for } k \in K \text{ and } f_k(x^*) \geq \tilde{f}_k,\ k \in K_1'' \cup K_3.$$

After transformation of the objective function $S(x)$ of the scalarizing problem E_1, using the inequalities (8), the following relation is obtained:

(9) $$S(x') = \max\Big[\max_{k\in K_1'}(\bar{f}_k - f_k(x'))/|f_k'|,\ \max_{k\in K_2}(f_k - f_k(x'))/|f_k'|\Big] + \max_{k\in K_1''}(f_k - f_k(x'))/|f_k'| =$$

$$= \max\Big[\max_{k\in K_1'}(\bar{f}_k - f_k(x^*)) + (f_k(x^*) - f_k(x')))/|f_k'|,$$

$$\max_{k\in K_2}(f_k - f_k(x^*)) + (f_k(x^*) - f_k(x')))/|f_k'|\Big] +$$

$$+ \max_{k\in K_1''}(f_k - f_k(x^*)) + (f_k(x^*) - f_k(x')))/|f_k'| <$$

$$< \max\Big[\max_{k\in K_1'}(\bar{f}_k - f_k(x^*))/|f_k'|,\ \max_{k\in K_2}(f_k - f_k(x^*))/|f_k'|\Big] + \max_{k\in K_1''}(f_k - f_k(x^*))/|f_k'| = S(x^*).$$

It follows from (9) that $S(x') < S(x^*)$ and $f_k(x^*) \geq \tilde{f}_k$, $k \in K_1'' \cup K_3$, which contradicts to (8). Hence x^* is a weak efficient solution of the multiple criteria integer problem (I).

Consequence. Theorem 1 is true for arbitrary values of f_k, $k \in K$.

The proof of this consequence follows from the fact that the proof of Theorem 1 does not assume any constraints on the values of the criteria f_k, $k \in K$.

Because the objective function of the scalarizing problem E_1 is nondifferentiable, one may solve the following equivalent mixed integer programming E_1':

(10) $$\min(\alpha + \beta)$$

subject to:

(11) $$\alpha \geq (\bar{f}_k - f(x))/|f_k'|,\ k \in K_1',$$

(12) $$\alpha \geq (f_k - f_k(x))/|f_k'|,\ k \in K_2,$$

(13) $$\beta \geq (f_k - f_k(x))/|f_k'|,\ k \in K_1'',$$

(14) $$f_k(x) \geq \tilde{f}_k,\ k \in K_1'' \cup K_3,$$

(15) $$x \in X_1,$$

(16) $$\alpha, \beta - \text{arbitrary}.$$

Problems E_1 and E_1' have the same feasible sets of the variables. The value of the objective functions of problems E_1 and E_1' are equal. This follows from the following assertion:

The scalarizing problem E_1' has four properties, that help to improve the dialogue with the DM, as with respect to the required from him/her information and with respect to the reducing of the waiting time for evaluation of new solutions also. The first property is connected with the required information from the DM. Instead of the aspiration levels of every criteria for the defining of the reference point [6, 7, 8], the

DM has to provide only changes in the criteria values of some of the criteria and the directions of change of the remaining criteria to specify the reference neighbourhood. The second property is that the current preferred solution is an initial feasible solution of the next integer problem E_1'. This facilitates the single criterion algorithms, especially the heuristic algorithms. The third property is that the feasible solutions of problem E_1' are near to the nondominated surface of the multicriteria integer problem (I). The application of heuristic algorithms to solve problem E_1' will lead to near (weak) nondominated solutions quickly, thus reducing the waiting time for the dialogue with the DM. The comparatively quick finding of more solutions for evaluation by the DM is important during the learning phase of the DM. The forth property of the problem E_1' is that with it the DM can realize the search strategy "no great benefit - little loss". The solutions obtained in the reference neighbourhood are comparatively close, which makes it easier for the DM to compare several solutions and choose the next preferred solution.

4. A reference neighbourhood interactive algorithm of multicriteria linear integer programming

A reference neighbourhood interactive algorithm solving multicriteria linear integer problems can be suggested on the basis of the scalarizing problems E_1'. The dialogue with the DM has been improved with respect to the information required from him/her; to the time when he/she is expecting a new solution; to the possibility for evaluation of more new solutions and to the learning possibilities of the specifics of the problem solved.

The basic steps of the algorithm are the following:

Step 1. An initial near (weak) nondominated solution of the multicriteria problem (I) is defined, setting $f_k = 1$, $k \in K$, $\bar{f}_k = 2$, $k \in K$, and solving approximately problem E_1'.

Step 2. Ask the DM to specify the reference neighbourhood of the current preferred solution defining desired changes in the values of some criteria, desired directions of change of other criteria and the values and/or directions of the eventual deterioration of the remaining criteria.

Step 3. Ask the DM to choose the type of the algorithm – exact or heuristic. If the DM selects an exact algorithm – go to *Step 5*.

Step4. Ask the DM to specify s – the maximal number of near (weak) nondominated solutions of the multicriteria problem (I), which can be stored in the set M_1. Solve the scalarizing problem E_1' with the help of an heuristic integer algorithm and present the set M_1 to the DM for evaluation and selection. In case the DM approves one solution as a current preferred solution of multicriteria problem (I) go to *Step 6*. If this solution is the last preferred solution – go to *Step 7*.

Step 5. Solve problem E_1'. Show the (weak) nondominated solution or near (weak) nondominated solution (if the computing process is interrupted) of the multicriteria problem (I) to the DM. In case the DM approves this solution as a current preferred solution of the multicriteria problem (I) go to Step *6*. If the solution is the last preferred solution – go to *Step 7.*

Step 6. If the DM wants to store the current preferred solution of the multicriteria problem (I) – check if it has been saved before, if not, add it to LIST – a set of stored preferred solutions – Go to *Step 2*.

Step 7. Does the DM want to compare the last preferred solutions of the multicriteria problem (I) with the solutions selected and stored in LIST – go to *Step 8*. If no – *Stop*. That is, the last preferred solution is the most preferred solution of the multicriteria problem (I).

Step 8. Show the set of solutions saved in LIST to the DM for comparison and selection of the most preferred solution of the multicriteria problem (I). *Stop*.

The proposed algorithm for solving multicriteria linear integer problems is a learning oriented [4] interactive algorithm and the DM controls the dialogue, the computations and the stopping conditions.

Problems of mixed integer linear programming (scalarizing problems) are solved in the interactive algorithm. The use of approximate algorithms [3, 5, 10, 11, 12] operating efficiently in a "narrow feasible area" and a known initial feasible integer solution enables the finding of good and in many cases – optimal solutions of the problems E_1'. The evaluation of more than one, even they be approximate (weak) nondominated solutions, enable the DM to learn faster with respect to the problems being solved.

The DM operates mainly in the criteria space, because in most of the cases the criteria have physical or economic interpretation and this enables the more realistic estimation and choice. The information required from the DM refers only to the defining of a reference neighbourhood of the current preferred solution.

5. Illustrative example

With the purpose to illustrate the interactive algorithm proposed the following multicriteria problem is solved. It is solved with the help of a developed small research software system and the DM is supposed to use all the possibilities that the algorithm provides.

$$\max f_1(x) = 5x_1 - x_2 + 2x_4,$$
$$\max f_2(x) = -x_1 + 2x_2 + x_3,$$
$$\max f_3(x) = 4x_2 - 8x_3 + 2x_4$$

under the constraints

$$-x_1 + 2x_2 + x_3 + 2x_4 \text{ I } 34,$$
$$2x_1 + x_2 - 3x_3 - x_4 \text{ I } 16,$$
$$3x_1 + 2x_2 + 4x_3 - x_4 \text{ I } 28,$$
$$x_1 + 6x_2 - x_3 + 4x_4 \text{ I } 43,$$
$$x_1, x_2, x_3, x_4 - \text{integer}.$$

Let us denote the feasible region for the integer variables by X_1.

In order to find an initial non-dominated solution, with the help of heuristic algorithm of mixed integer programming a scalarizing problem of E_1 type is solved, at apriori set $f_k = 1$, $k \in K$, $\bar{f}_k = 2$, $k \in K$, and $K_1' = \{1, 2, 3\}$:

$\min \alpha,$

$\alpha \ \mathrm{i}\ 2 - 5x_1 + x_2 - 2x_4,$

$\alpha \ \mathrm{i}\ 2 + x_1 - 2x_2 - x_3,$

$\alpha \ \mathrm{i}\ 2 - 4x_2 + 8x_3 - 2x_4,$

$x \in X_2,$

α – arbitrary.

The values of the criteria for the solution obtained are: $f_1 = 10$, $f_2 = 10$, $f_3 = 10$. Let us assume that the DM would like to improve the first criterion and sets only the reference point $\bar{f}_1 = 15$, $K_1' = \{1\}$ for it; he agrees to deteriorate the second criterion, not defining by what value, $K_2 = \{2\}$, and to improve the third criterion, not defining any certain values, $K_1'' = \{3\}$ and $K_3 = \varnothing$. The DM would like to see integer solutions at this iteration, but not more than 3, i. e., $s = 3$. A problem of E_1' type is formed:

$\min(\alpha + \beta),$

$\alpha \ \mathrm{i}\ (15 - 5x_1 + x_2 - 2x_4)/10,$

$\alpha \ \mathrm{i}\ (10 + x_1 - 2x_2 - x_3)/10,$

$\alpha \ \mathrm{i}\ (10 - 4x_2 + 8x_3 - 2x_4)/10,$

$4x_2 - 8x_3 + 2x_4 \ \mathrm{i}\ 10,$

$x \in X_1,$

α, β – arbitrary.

With the help of an heuristic algorithm of mixed integer programming problems the following solution is found: $f_1 = 14$, $f_2 = 8$, $f_3 = 24$.

The value of the second criterion does not satisfy the DM and he/she chooses to continue with aspiration level of the second criteria – $\bar{f}_2 = 12$, $K_1' = \{2\}$. He agrees to deteriorate the first and third criteria, i. e. $K_2 = \{1, 3\}$. A scalarizing problem of E_1' type is solved and solution is found.

$\min \alpha,$

$\alpha \ \mathrm{i}\ (12 + x_1 - 2x_2 - x_3)/8,$

$\alpha \ \mathrm{i}\ (24 - 4x_2 + 8x_3 - 2x_4)/24,$

$\alpha \ \mathrm{i}\ (14 - 5x_1 + x_2 - 2x_4)/14,$

$x \in X_1,$

α – arbitrary.

The values of the criteria for the solution obtained are $f_1=11$, $f_2=10$ and $f_3=10$. This solution does not satisfy the DM either and he sets new conditions for the criteria: to improve the first and the third criteria, $K_1'' = \{1, 3\}$, the second one may be deteriorated by 1 unit, $\tilde{f}_2 = 9$, $K_3 = \{2\}$. A following scalarizing problem of E_1' type is solved with the help of an exact algorithm of mixed integer programming.

$$
\begin{aligned}
&\min \beta,\\
&\beta \geq (11 - 5x_1 + x_2 - 2x_4)/11,\\
&\beta \geq (10 - 4x_2 + 8x_3 - 2x_4)/10,\\
&5x_1 - x_2 + 2x_4 \geq 11,\\
&4x_2 - 8x_3 + 2x_4 \geq 10,\\
&-x_1 + 2x_2 + x_3 \geq 9,\\
&x \in X_1,\\
&\beta \text{ – arbitrary.}
\end{aligned}
$$

The solution found is: $f_1 = 14$, $f_2 = 9$, $f_3 = 18$.

The DM selects this solution as the most satisfactory for him/her. With this the operation of algorithm is brought to an end.

6. Conclusion

A learning-oriented interactive algorithm is proposed based on the reference neighbourhood approach to solve multicriteria linear integer programming problems. The scalarizing problems E_1' provide the opportunity to improve the dialogue with the DM with respect to several features:

– according to DM's wish, he/she may set different type and different quantity of information at each iteration;

– the time during which he/she is expecting solutions for evaluation and choice is reduced;

– his/her possibilities for learning the specifics of the multiple criteria integer problems being solved can be increased.

These features of interactive algorithm characterise it as an appropriate and user-friendly algorithm solving multicriteria linear integer programming problems.

7. References

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Интерративный алгоритм отправной области для решения задач многокритериального линейного целочисленного программирования

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(Р е з ю м е)

Представлен итерративный алгоритм, ориентированный к обучении, который предназначеный для решения линейных задач многокритериального целочисленного программирования. При каждой итерации лицо, принимающее решение (ЛПР), задает свои локальные предпочитания в виде желаемых изменений стоимостей некоторых из критерий, желаемых направлений изменения других критерий и направления и/или стоимости евентуального ухудшения остальных критерий. Локальные предпочитания ЛПР определяют отправную область. На основе этой отправной области формулируется целочисленная скаларизирующая задача.