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Capacity and Maximal and Minimal Network Flow with Additional Linear Equalities

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1. Introduction

The theoretical and applied interest towards network flows in the last decades is supported by their significant use in modern information technologies.

The most widely spread classical network flows are those of Ford and Fulkerson [1, 2, 3], in which the capacity is defined on each arc of the network. In the network flows investigated in [6, 7] besides the constraints on the separate arcs, some inequalities are used, including separate sets of the arc flow functions as variables.

A class of network flows is suggested in [6, 7], where the arc capacities are replaced by linear inequalities, and the variables are subsets of arc flow functions. This class of network flows is called a linear flow.

A class of network flows is defined and studied in [8, 9], where the capacity of the separate arcs is replaced by one [8], or by a set [9] of additional linear equalities with arbitrary coefficients in the left side and nonzero coefficients – in the right. The latter of these classes of network flows is called an ALE-flow. An ALE-flow is stated in [9] and different problems of its existence are studied.

The present paper concerns the definition of the ALE-flow capacity and its dependence on the maximal and minimal values of the same flow.

2. Definition of the capacity of the ALE-flow and of its lower and upper limit

Let the graph $G(N, U)$ be defined by a set of nodes N and a set of arcs U. The set M contains the indices of all the simple oriented paths from the source S towards the sink t, in which there are no cycles and each path $\mu\in M$ includes the separate nodes and arcs only once [1, 2]. $U(\mu)$ will denote the set of arcs of the path with an index $\mu\in M$:

(1) $$U = \bigcup_{\mu\in M} U(\mu).$$

The network flow with additional linear equalities (ALE-flow) will be defined by the following constraints: for each

(2) $$f(x, N) - f(N, x) = \begin{cases} v, \text{ if } x = s; \\ 0, \text{ if } x \neq s, t; \\ -v, \text{ if } x = t; \end{cases}$$

(3) $$\sum_{(x,y)\in D_i} b_i(x, y)\, f(x, y) = C_i\,;\; i \in I;$$

(4) $$f(x, y) \geq 0;\; (x, y) \in U;$$

where I is the set of the indices of the linear equalities (3); $C_i \geq 0$ are rational non-negative numbers; $D_i \subseteq U$ are subsets of the division of U, i.e., for each $i, j \in I$, it is true that

(5) $$D_i \cap D_j = \varnothing, \quad \bigcup_{i\in I} D_i = U;$$

$\varnothing$ – an empty set; v and f are a flow and an arc flow function respectively, for which

(6) $$v \geq 0; \quad f(x, y) \geq 0;\; (x, y) \in U;$$

$$f(x, N) = \sum_{y\in\Gamma^1(x)} f(x, y); \quad f(N, x) = \sum_{y\in\Gamma^{-1}(x)} f(y, x);$$

(7) $$(N, s) = (t, N) = \varnothing;$$

(8) $$b_i(x, y) \begin{cases} \in R', \text{ if } (x, y) \in D; i \in I; \\ = 0 \text{ otherwise}; \end{cases}$$

R' is a set of all nonzero rational numbers, $\Gamma^1(x)$ and $\Gamma^{-1}(x)$ are an image and inverse image of x into N.

It is assumed that each subset D_i, $i \in I$, is contained in the arcs set (1) of all the simple oriented paths, i.e.,

(9) $$\{\bigcup_{\mu\in M} U(\mu)\} \cap D_i = D_i.$$

The ALE-flow is defined in relations from (2) up to (4) in the form of arc-nodes. Further on the main form for the representation of this flow will be arcs-paths [1]

The following denotations are introduced:

M_i – a set of these paths in M, in which at least one arc from D_i is contained;

$M_i = \{\mu | \mu \in M;\; U(\mu)\} \cap D_i \neq \varnothing\}$, $i \in I$; $U_i(\mu)$ – a set of arcs on the path μ, which are contained in D_i, i.e.,

(10) $$U_i(\mu) = U(\mu) \cap D_i;\; i \in I;\; \mu \in M;$$

$B_i(\mu)$ – the sum of the coefficients, corresponding to $U_i(\mu)$, at that for each $i \in I$ and $\mu \in M$, the following is satisfied:

(11) $$B_i(\mu) = \begin{cases} \sum_{(x,y)\in U_i(\mu)} b_i(x, y), \text{ if } U_i(\mu) \neq \varnothing; \\ 0 \text{ otherwise}; \end{cases}$$

$U(\mu)$ – a flow, corresponding to the path $\mu \in M$ [1], for which

(12) $$v(\mu) \begin{cases} \leq f(x, y), \text{ if } (x, y) \in U(\mu); \\ = 0 \text{ otherwise}; \end{cases}$$

(13) $$v = \sum_{\mu \in M} v(\mu);$$

α_μ – relative coefficients, for which

(14) $$\alpha_\mu = \begin{cases} v(\mu)/v, & \text{if } v>0;\ \mu \in M; \\ 0 & \text{otherwise;} \end{cases}$$

$$\sum_{\mu \in M} \alpha_\mu = 1;\ 0 \le \alpha_\mu \le 1,\ \text{if } v>0 \text{ and } \mu \in M.$$

$|M| = m$ – number of the elements in the set M, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ – flow realization.

$B(\alpha, i)$ is an $L(i)$-factor [9], for which

(15) $$B(\alpha, i) = \sum_{\mu \in M_i} \alpha_\mu B_i(\mu);\ i \in I.$$

The following relations are proved in [9] for the ALE-flow thus defined: for each realization α and $i \in I$

(16) $$\sum_{(x,y)\in D_i} b_i(x, y)\, f(x, y) = \sum_{\mu \in M_i} v(\mu) B_i(\mu) = vB(\alpha, i).$$

The following four values – C_i^1, C_i^2, C^1 and C^2, play an important role in the definition of the ALE-flow capacity

(17) $$B_i^1 = \min_{\mu \in M_i} B_i(\mu);\quad B_i^2 = \max_{\mu \in M_i} B_i(\mu);$$

(18) $$C_i^1 = \begin{cases} C_i/B_i^1, & \text{if } B_i^1>0; \\ +\infty & \text{otherwise;} \end{cases}$$

(19) $$C_i^2 = \begin{cases} C_i/B_i^2, & \text{if } B_i^2>0; \\ 0 & \text{otherwise.} \end{cases}$$

The values C^1 and C^2 are defined by the following two linear programming problems:

(20) $$C^1 = \sum_{\mu \in M} v(\mu) \to \max,$$

under the constraints: for each $i \in I$

(21) $$\sum_{\mu \in M_i} v(\mu) B_i(\mu) = C_i;$$

(22) $$v(\mu) \ge 0;\quad \mu \in M;$$

(23) $$C^2 = \sum_{\mu \in M} v(\mu) \to \min,$$

subject to (21) and (22).

When comparing the values from (17) upto (23) it follows that C_i^1 and C_i^2 define the upper and lower limit of the flow v respectively in the presence of only one equality from (3) with an index $i \in I$, and the parameters C^1 and C^2 show the same limits of the flow for a set of all the equalities of (3) with indices from I.

Definition 1. The values C^1 and C^2 from (20) and (23) will be called upper and lower limit of the capacity of the ALE-flow from (2) upto (4).

Lemma 1. If there exists an ALE-flow from (2) upto (4), it is true for the flow v that:

(24) $$C^2 \le v \le C^1.$$

Proof. The conformity of relations (13), (15) and (16) with those of (20) upto (22) show the validity of the right inequality in (24), and the comparison of (13), (15) and (16) with the relations from (21) upto (23) – the validity of the left one from these inequalities.

At a linear equality $I = \{i\}$ it is not necessary to solve problems from (20) upto (23) – for C_i^1 and C_i^2 the use of the values from (17) upto (19) is sufficient. In this case $C^1 = C_i^1$ and $C^2 = C_i^2$.

In case there exists an ALE-flow, the following consequences can be derived from Lemma 1.

Consequence 1.1. If for each $i \in I$ and $\mu \in M_i$

$$B_i(\mu) = 0 \text{ and } C_i = 0,$$

then

(25) $$0 \le v \le +\infty.$$

Consequence 1.2. In case there exists one realization only of the flow, satisfying relations from (2) upto (4), and

$$0 \le v \le +\infty$$

then

(26) $$v = C^1 = C^2.$$

Consequence 1.3. If for every $i \in I$ and $\mu \in M_i$

$$B_i(\mu) > 0 \text{ and } C_i = 0,$$

then

(27) $$v = C^1 = C^2 = 0.$$

Consequence 1.4. If for each $i \in I$ and $\mu \in M_i$

$$B_i(\mu) < 0 \text{ and } C_i = 0,$$

then

(28) $$v = C^1 = C^2 = 0.$$

Consequence 1.5. If for every $i \in I$

$$C_i = 0$$

then

(29) $$0 \le v \le C^1.$$

The following two confirmations follow directly from inequalities (24) of Lemma 1.

Confirmation 1. If there exists an ALE-flow from (2) upto (4), its maximal value v_{max} is equal to the upper limit of the capacity C^1.

Confirmation 2. In case there exists an ALE-flow from (2) upto (4) its minimal value v_{min} is equal to the lower limit of the capacity C^2.

Confirmation 1 can be regarded as an analogue for the ALE-flow of the mincut-maxflow theorem of Ford and Fulkerson [1] for the equality of the minimal cut and maximal flow in the classical network flow. For an ALE-flow the statement of this relation is different and according to confirmation 1 it defines the equality of the upper limit of the capacity C^2 of the maximal ALE-flow.

Unlike the other classes of flows, according to Lemma 1, there may exist a lower bound of the capacity C^1 with a positive value, equal to the minimal possible value of the flow v_{min}. This can also be regarded as a specific mincut-maxflow analogue of the fundamental theorem of Ford and Fulkerson.

3. Capacity of ALE-flow at different subsets of linear equalities

We will consider the cases when the defining of the upper and lower limit of the capacity is realized not only by all the equalities of (3), but also using different subsets of these equalities.

Let the family of subsets $\{I(r) \mid r \in G\}$ be defined in the set of indices I such that

(30) $$I = \bigcup_{r \in G} I(r);\ I(r) \subseteq I;\ r \in G$$

where G is a set of the indices of the family of the subsets in I.

Further on we shall discuss only these subsets in I, for which for each arbitrary index $r \in G$ and $p \in G$ the subsets of arcs corresponding to $I(r)$ and $I(p)$ block the whole set of paths from S towards t, i.e., for each $\mu \in M$ it is true that

$$U(\mu) \cap \{\bigcup_{i \in I(r)} \bigcup_{\mu \in M_i} U_i(\mu) \neq \emptyset\};$$

$$U(\mu) \cap \{\bigcup_{i \in I(p)} \bigcup_{\mu \in M_i} U_i(\mu) \neq \emptyset\}.$$

In an analogous way, as in relations from (20) up to (23) it can be written: for each $r \in G$

(31) $$C^1(r) = \sum_{\mu \in M} v(\mu) \to \max$$

under constraints

(32) $$\sum_{\mu \in M_i} v(\mu)\, B_i(\mu) = C_i;\ \mu \in M_i;\ i \in I(r);$$

(33) $$v(\mu) \geq 0;\ \mu \in M_i;\ i \in I(r);$$

(34) $$C^2(r) = \sum_{\mu \in M} v(\mu) \to \min$$

and constraints (32) and (33).

Lemma 1 has the following form for the subset $I(r)$:

(35) $$C^2(r) \leq v \leq C^1(r).$$

Five consequences can be derived from (32), analogous to those from (25) up to (29), for $I(r)$, $C^1(r)$ and $C^2(r)$.

Theorem 2. If for two subsets $I(r)$, $r \in G$, and $I(p)$, $p \in G$, the respective ALE-flows exist and it is true that

(36) $$I(p) \subset I(r);$$

then

(37) $$C^1(r) \leq C^1(p);$$

(38) $$C^2(r) \geq C^2(p).$$

Proof. The parameters $C^1(p)$ and $C^2(p)$ can be defined similarly to those for $r \in G$, and exactly:

(39) $$C^1(p) = \sum_{\mu \in M} v(\mu) \to \max;\ C^2(p) = \sum_{\mu \in M} v(\mu) \to \min.$$

satisfying constraints (32) and (33), but for each $i \in I(p)$.

A. It is assumed that

(40) $$C^1(r) > C^1(p).$$

Since the objective functions (31) and the first one in (39) are equal for $C^1(r)$ and $C^1(p)$, and it follows from (36) that the equalities $I(p)$ are a part of $I(r)$, but do not coincide with them, assumption (40) means that there could be found a plan $\{v(\mu) \mid \mu \in M\}$ for which the equalities from $I(r)$ and $I(p)$ are satisfied and for which

$$\sum_{\mu \in M} v(\mu) > C^1(\mu). \tag{41}$$

But this contradicts to the first one in relations (39) and shows the impossibility of (40).

B. Let the graph $G(N, U)$ be defined in such a way that

$$\{\mu',\mu''\} = M;\ I(p) = \{i\};\ I = I(r) = \{i, j\};$$

$$U_i(\mu') = \{(x, y)\};\quad U_i(\mu'') = \{(x, z)\};$$

$$U_j(\mu') = \{(y, t)\};\quad U_j(\mu'') = \{(z, t)\}.$$

Then it follows from (3), (11), (15), and (16) that for each $i \in I$

$$v(\mu')B_i(\mu') + v(\mu'')B_i(\mu'') = C_i.$$

Since it follows from (41) that $v(\mu'')=v-v(\mu')$, then

$$vB_i(\mu'') + v(\mu')(B_i(\mu')-B_i(\mu'')) = C_i. \tag{42}$$

In a similar way for $j \in I$ it will be obtained:

$$vB_j(\mu'') + v(\mu')(B_j(\mu')-B_j(\mu'')) = C_j. \tag{43}$$

If it is assumed that

$$B_i(\mu') = B_i(\mu'') = 0 \text{ and } C_i = 0,$$

$$B_j(\mu') = B_j(\mu'') = b_j > 0 \text{ and } C_j > 0,$$

then from (42), (43) and the above assumptions for equality from $\{i\} = I(p)$, it will be obtained:

$$v_{\max} = +\infty = C^1(p),$$

and the simultaneous satisfying of the equalities from $\{i, j\} = I(r)$ leads to

$$v_{\max} = C_j / b_j = C^1(r).$$

Hence the following strict inequality is true:

$$C^1(r) < C^1(p). \tag{44}$$

If it is assumed that $C_j=0$ or $b_j \to +\infty$, then we shall reach the equality:

$$C^1(r) = C^1(p). \tag{45}$$

From the invalidity of (40) and the cases (44) and (45), the relation (37) follows. In a similar way (38) can be proved.

The following useful results can be derived from this theorem.

Definition 2. If for two groups of indices $I(r)$, $r \in G$, and $I(p)$, $p \in G$, it is true that

$$I(p) \subseteq I(r); \tag{46}$$

$$C^1(p) = C^1(r), \tag{47}$$

then the set $I(r)$ will be called r-minimal.

If $I(r^*)=I$, then the class of equalities $I(r^*)$ will be called complete, the r^*-minimal class $I(p)$ will be called just minimal.

Consequence 2.1. If at least one of the two subsets $I(r_1)$ or $I(r_2)$ is r-minimal, then ALE with constraints from $I(r_3)$, for which

$$I(r_3) = I(r_1) \cup I(r_2) \subseteq I(r_3), \tag{48}$$

is also r-minimal.

Proof. It is assumed that $I(r_1)$ is r-minimal, and hence

$$C^1(r) = C^1(r_1). \tag{49}$$

It follows from (48) that

$$I(r_1) \subseteq I(r_3).$$

The relation above given and Theorem 2 lead to

(50) $$C^1(r_1) \geq C^1(r_3).$$

From (49) and (50) it follows that

(51) $$C^1(r) \subseteq C^1(r_3).$$

It can be written from condition (48) that

$$I(r_3) \subseteq I(r)$$

and according to Theorem 2

(52) $$C^1(r_3) \geq C^1(r).$$

The r-minimality of the equalities subset $I(r_3)$ follows from (51) and (52).

Due to the arbitrary choice of r_1 the following can be proved in a deductive way.

Sequence 2.2. If at least one of the sets $I(r_j)$, $j=1, \ldots, n$, for which

(53) $$I(r_1) \cup I(r_2) \cup \ldots \cup I(r_n) = I(r_m),$$

is r-minimal, then the set $I(r_m)$ is also r-minimal.

Similar relations can be obtained for the upper limit of the capacity $C^2(r)$, $r \in G$.

Definition 3. If for two groups of indices $I(r)$, $r \in G$, and $I(p)$, $p \in G$, (46) is true and

(54) $$C^2(p) = C^2(r),$$

then the set $I(p)$ will be called r-maximal, the r^*-maximal class $I(p)$ is called simply maximal.

Consequence 2.3. If at least one of the two subsets $I(r_1)$ or $I(r_2)$ is r-maximal, then the ALE-flow with constraints from $I(r_3)$, for which (48) is true, is also r-maximal.

Consequence 2.4. If at least one of the sets $I(r_j)$, $j=1, \ldots, n$, for which (53) is true, is r-maximal, then the set $I(r_m)$ is also r-maximal.

The last two consequences can be proved with the help of the same logical scheme as consequences 2.1 and 2.2.

4. Conclusion

1. A nonclassical network flow with additional linear equalities – an ALE-flow [9] has been studied in the paper. The arc capacities in the flow of Ford and Fulkerson in it are replaced by a set of linear equalities, the left part of which is a sum of multiplied by coefficients arc flow functions, and the right one consists of non-negative coefficients.

2. A method is suggested for the definition of the upper and lower limits of the capacity of the ALE-flow by linear equalities. The coefficients in the left and right side of the linear equalities play a significant part. A relation has been proved, indicating that if there exists an ALE-flow, its value is found between these two limits.

3. Theorems have been proved verifying that the maximal ALE-flow is equal to the upper limit of the capacity, and the minimal ALE-flow – to the lower limit of the capacity. They can be regarded as specific analogues of the famous mincut-maxflow and maxcut-minflow theorem of the classical network flow.

4. Results have been obtained for the behaviour of the upper and lower limits of the ALE-flow capacity using different, mutually contained one in another subsets of a set of linear equalities.

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Пропускная способность и максимальный и минимальный сетевой поток с дополнительными линейными равенствами

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(Р е з ю м е)

В работе исследован неклассический сетевой поток с дополнительными линейными равенствами – ДЛР-поток. В нем дуговые пропускные способности в потоке Форда-Фулькерсона заменены множеством линейных равенств, левая часть которых является суммой дуговых потоковых функций, помноженных на коэффициенты, а правая часть состоит из неотрицательных коэффициентов.

Предложен способ определения верхней и нижней границ пропускной способности ДЛР-потока с помощью линейных равенств. В нем существенное значение имеют коэффициенты левой и правой части линейных равенств. Доказана зависимость, которая показывает, что если существует ДЛР-поток, то его значение находится между двумя этими границами.

Доказаны теоремы, согласно которых максимальный ДЛР-поток равен верхней границе пропускной способности, а минимальный ДЛР-поток – нижней границе пропускной способности. Их можно рассматривать как аналог известных ***mincut-maxflow*** теоремы и ***maxcut-minflow*** теоремы классического сетевого потока.

Получены результаты о поведении верхней и нижней границ пропускной способности ДЛР-потока при исспользовании разных, содержающихся одно в другом подмножеств множества линейных равенств.