БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ . BULGARIAN ACADEMY OF SCIENCES

ПРОБЛЕМИ НА ТЕХНИЧЕСКАТА КИБЕРНЕТИКА И РОБОТИКАТА, 49
PROBLEMS OF ENGINEERING CYBERNETICS AND ROBOTICS, 49

София . 2000 . Sofia

# A Multicommodity Network Flow with Inverse Linear Constraints

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## 1. Introduction

In many application areas – engineering, communications, logistics, manufacturing, transportation, different non-homogeneous commodities are distributed over the same underlying network. Usually the separate commodities share common arc capacities that restrict the integrated flow of the gross commodities on the arc. Furthermore, there exists a mutual interaction between the commodities. The generic multicommodity flow is a comparatively complex generalization of the standard single-commodity flow [1, 2, 4]. Another more general class of single-commodity flow is introduced and investigated in [3] – a network flow with inverse linear constraints (ILC-flow). The values of this flow are bounded down by linear inequalities with real non-zero coefficients. Both generalizations do not have the specific properties as the standard single-commodity flow. The respective problems do not necessary provide integer flows notwithstanding the input data - the supply/demand and the capacity, is integer valued . Still they are linear programs with special structures that allow the use of the decomposition approach.

The present paper discusses a multicommodity network flow with inverse linear constraints. The results obtained for the ILC-flow are extended to this flow. The network properties of the investigated flow, although reduced, are exploited considerably.

## 2. Definition of a multicommodity ILC-flow

Let several non-homogeneous flows with indices $k \in K$ be given on the directed connected graph $G(N,U)$, where $N$ is the set of nodes $|N| = n$ and $U$ is the set of arcs $|U| = m$. For each commodity the flow is preserved at every intermediate node and the values of these flows $f^k(x, y)$ on the separate arcs $(x, y) \in U$ are bounded down by inverse linear constraints. At a multiterminal case the set of sources $S=\{s^k | k\in K\}$ and the set of sinks $T=\{t^k | k\in K\}$ are reduced to single source $s$ and single sink $t$, and the respective arcs are added. Further, this two-terminal case is concerned.

The multicommodity ILC-flow is defined in the following way: for each $k\in K$, $x\in N$ and $i\in I$

(1)
$$f^k(x,N)-f^k(N,x)=\begin{cases} v^k \text{ if } x=s \\ 0 \text{ if } x\neq s,t \\ -v^k \text{ if } x=t, \end{cases}$$

(2)
$$\sum_{k\in K}\ \sum_{(x,y)\in D_{i,k}} b^k_i(x,y)\, f^k(x,y)\geq c_i,\ i\in I,$$

(3)
$$f^k(x,y)\geq 0;\ (x,y)\in U,$$

where $c_i$, $i\in I$ are real nonnegative numbers, $D_{i,k}$, $i\in I$ – an arbitrary subset of $U$, such that $D_{i,k}\subseteq U$ and $\bigcup_{i\in I}\ \bigcup_{k\in K} D_{i,k}=U$, $\varnothing$ – an empty set,

$$b^k_i(x,y)\ \begin{cases} \in R', \text{ if } i\in I \text{ and } (x,y)\in D_{i,k}, \\ =0 \text{ otherwise;} \end{cases}$$

$R'$ – a set of real non-zero numbers.

The investigated flow, defined by (1)–(3), differs from the standard multicommodity flow with lower bound of the capacity. This flow is defined by (1), (3) and the following constraints:

(5)
$$\sum_{k\in K} f^k(x,y)\geq c(x,y);\ (x,y)\in U.$$

3. Characteristics of the multicommodity ILC-flow

Further on the feasible multicommodity ILC-flows

(6)
$$v=\sum_{k\in K} v^k$$

and the minimal multicommodity ILC-flows

(7)
$$v=\min\sum_{k\in K} v^k$$

will be considered.

The necessary and sufficient conditions for the existence of the single commodity ILC-flow, if the values $\{c_i\}$ are positive real numbers, are derived in [3]. The proof for the existence of multicommodity ILC-flow, on the same assumption, may be accomplished in similar way.

A minimal multicommodity ILC-flow $\{v^k\}$ is determined by the following linear programming problem:

(8)
$$\min\sum_{k\in K} v^k$$

subject to the constraints (1)–(3).

**Lemma 1.** If $\{f^k\}$ is a multicommodity ILC-flow between $s$ and $t$ with values $\{v^k\}$, and $(X, \bar{X})$ is an arbitrary cut, separating the source from the sink, then for each $k\in K$

(9) $$v^k = f^k(X, \bar{X}) - f^k(\bar{X}, X),$$

The proof can be constructed by successively summing the equations of flow conservation (1) in all nodes $x\in X$ ■

At cutting sets the functions g and w are used instead of f and v.

The capacity $c(r)$ of the cutting set $U(r)$ in case of multicommodity ILC-flow is determined, as is for the single-commodity ILC-flow, by the following linear programming problem:

(10) $$c(r) \rightarrow \min \sum_{k\in K} w^k$$

subject to constraints

(11) $$g^k(x, N) - g^k(N, x) = 0,\ x\in N(r),\ k\in K,$$

(12) $$g^k(X(r), \bar{X}(r)) - g^k(\bar{X}(r), X(r)) = w^k,\ k\in K,$$

(13) $$g^k(Y(r), \bar{Y}(r)) - g^k(\bar{Y}(r), Y(r)) = w^k,\ k\in K,$$

(14) $$\sum_{k\in K} \sum_{(x,y)\in D_{i,k_0}(r)} b^k_i(x, y)\, g^k(x, y) \geq c_i,\ i\in I(r),$$

(15) $$g^k(x, y) \geq 0,\ (x,y)\in U'(r),$$

where

(16) $$I(r) = \{i | U(r)\cap D_{i,k} \neq \varnothing,\ i\in I,\ k\in K\};$$

(17) $$D'_{i,k}(r) = \begin{cases} U(r)\cap D_{i,k} \text{ if } i\in I(r),\ k\in K, \\ \varnothing \text{ otherwise;} \end{cases}$$

(18) $$D''_{i,k}(r) = \begin{cases} D_{i,k}\setminus D'_{i,k}(r) \text{ if } i\in I(r),\ k\in K, \\ \varnothing \text{ otherwise;} \end{cases}$$

(19) $$D_{i,k} = D'_{i,k}(r) \cup D''_{i,k}(r);$$

(20) $$D'_{i,k_0} = D_{i,k}\setminus D''_{i,k}(r) \cup \Big( \bigcup_{k\in K\setminus k_0} D'_{i,k}(r) \Big);$$

$X(r)$ and $Y(r)$ are cutting sets;

(21) $$N(r) = Y(r)\cap \bar{X}(r),\ U'(r) = (Y(r), \bar{X}(r)) \cup (\bar{X}(r), Y(r)),\ \Delta U(r) = U'(r)\setminus U(r).$$

The existence of a solution of this linear programming problem may be proved in an analogous way as for the single commodity ILC-flow considering that all flows $g^k$, $k\in K$, are nonzeroes and down bounded.

The following theorem is essential for elucidation of the cutting set capacity significance in multicommodity ILC-flow optimality.

**Theorem 2.** If in a network with multicommodity ILC-flow there are given two cutting sets $U(r)$ and $U(p)$, such that

(22) $$U(r) \subseteq U(p),$$

then

(23) $$c(p) \ \mathrm{i} \ c(r).$$

The proof may be constructed in a similar way as for the single-commodity ILC-flow.

**Definition 1.** A cutting set $U(r^*)$, for which

(24) $$U(r^*)=U$$

is a full cutting set.

**Definition 2.** If for two cutting sets $U(r)$ and $U(p)$

(25) $$U(p) \subseteq U(r) \text{ and } c(p)=c(r),$$

then the cutting set $U(p)$ is called $r$-maximal; $r^*$– maximal cutting set is called maximal.

**Lemma 3.** For each network the minimal value of a multicommodity ICL-flow from $s$ to $t$, $v^k_{\min}$, is equal to the capacity of the full cutting set $c(r^*)$, that is

(26) $$\min \sum_{k \in K} v^k = c(r').$$

*P r o o f.* For the multicommodity ICL-flow the following relations hold, too:

(27) $$f^k(N, s)=0, \; f^k(t, N)=0, \; k \in K.$$

At $r=r^*$ the following relations hold:

(28) $$N(r)=N \backslash (s \cup t), \; I(r)=I; \; U'(r)=U, \; \Delta U(r)=\varnothing;$$

(29) $$g^k(X(r), \bar{X}(r))=g^k(s, N), \; k \in K;$$

(30) $$g^k(Y(r), \bar{Y}(r))=g^k(N, t), \; k \in K.$$

Therefore, both linear programming problems – for determination of $\sum_{k \in K} v^k$ and $c(r^*)$ have one and the same forms and constraints■

The next theorem follows from Lemma 3 and Definition 2.

**Theorem 4.** (Theorem for the minimal multicommodity ICL-flow and the capacity of the maximal cutting set). For each network the minimal value of the multicommodity ICL-flow from $s$ towards $t$ is equal to the capacity of the maximal cutting set.

**Lemma 5.** If $\{v^k\}$ is a multicommodity ILC-flow on a network the sufficient condition for its minimality is the presence of a cutting set with index $r$, not coinciding with the full cutting set with index $r^*$, for which it holds:

(31) $$\sum_{k \in K} v^k = c(r).$$

*P r o o f.* The sufficient condition (31) for minimality of the multicommodity ILC-flow follows from Theorem 2 and Lemma 3 ■

On the other hand relation (31) is a sufficient condition for maximality of the cutting set with index $r$.

**Lemma 6.** If $\{f^k\}$ is a multicommodity ILC-flow from $s$ towards $t$ with value $\sum_{k \in K} v^k$ and $U(r)$ is an arbitrary cutting set separating $s$ from $t$, then

(32) $$\sum_{k\in K} v^k = \sum_{k\in K} f^k(X(r), \bar{X}(r)) - f^k(\bar{X}(r), X(r))) \geq c(r),$$

(33) $$\sum_{k\in K} v^k = \sum_{k\in K} f^k(Y(r), \bar{Y}(r)) - f^k(\bar{Y}(r), Y(r))) \geq c(r).$$

*Proof.* The arc sets $(X(r), \bar{X}(r))$ and $(Y(r), \bar{Y}(r))$ are cuts under construction. Therefore the equalities in (32) and (33) follow from Lemma 1. From $U(r)\subseteq U$ and Theorem 2 the following relation holds

$$c(r) \leq c(r^*).$$

This result and (26) lead to the strict inequalities in (32) and (33)■

From the exposition upto here it follows that all results for the capacity of the single-commodity ILC-flow can be extended on the multicommodity ILC-flow with a given in analogous way capacity.

Denote by $c(r, K)$ the capacity $c(r)$ of the cutting set at multicommodity ILC-flows $K$ on a network.

**Theorem 7.** At fixed constraints (14), if

(34) $$K_1\subseteq K,$$

then

(35) $$c(r, K_1) \leq c(r, K).$$

*Proof.* Let

(36) $$K = K_1 \cup \{k^*\}.$$

After the determination of $c(r, K_1)$ and $c(r, K)$ the values of $g^k$ and $w^k$ are denoted respectively by $g_0$, $w_0$ and $g_1$, $w_1$.

a) Assume

(37) $$\sum_{k\in K_1} w_1 > \sum_{k\in K} w_0.$$

The multicommodity ILC-flow realization $g_1$, $w_1$ at $k\in K_1$ is extended to $g_2^k$, $w_2^k$ at $k\in K$ in the following way: for each $(x, y)\in U$ and $k\in K$

(38) $$g^k_2(x, y) = \begin{cases} 0 \text{ if } k = k', \\ g_1(x, y) \text{ otherwise.} \end{cases}$$

The values $g^k_2(x,y)$ are a feasible realization for determination of $c(r, K)$. From (10) to (21), (36) and (37), it follows that

(39) $$\sum_{k\in K} w_1^{\ k} = \sum_{k\in K} w_2^{\ k} > \sum_{k\in K} w_0^{\ k}.$$

In (39) the inequality means that a realization can be found, which is better than the optimal one. This is a contradiction.

b) The following case is examined: for each $i\in I$, $(x, y)\in U(r)$ and $k\in K$

(40) $$D_{i,k}(r) = D'_{I,k}(r),\ b_i^k(x, y) = b_i^{k^*}(x, y)$$

From (10)–(21) and (40) it follows

(41) $$c(r, K_1) = c(r, K).$$

c) Let $I(r)=\{1, 2\}$ and for each $k \in K$

(42) $$D'_{1,k}(r) \begin{cases} \neq \varnothing, \text{ if } k = K_1, \\ = \varnothing \text{ otherwise;} \end{cases}$$

(43) $$D'_{2,k}(r) \begin{cases} \neq \varnothing, \text{ if } k = k^*, \\ = \varnothing \text{ otherwise.} \end{cases}$$

(44) $$\text{If } c_1 > c_2,$$

then from (10)–(21) and (42)–(44) it is obtained

(45) $$c(r, K_1) < c(r, K).$$

Because of the assumption (39), the incorrectness of (37) and the examples for which (41) and (44) hold, (45) follows. By induction the relation (45) may be proved for an arbitrary $K_1 \subset K$ ■

**Definition 3.** If for an arbitrary cutting set $U(r)$ and two multicommodity ILC-flows $K_1$ and $K$:

(46) $$K_1 \subseteq K,$$

(47) $$c(r, K_1) = c(r, K)$$

then the multicommodity ILC-flow $K_1$ is called $(r, K)$-minimal.

$(r^*, K)$-minimal multicommodity ILC-flows are denoted as $K$-minimal.

From Definition 4 it follows that for each multicommodity ILC-flow there exists at least one $(r, K)$-minimal multicommodity ILC-flow.

**Theorem 8.** If at least one of the multicommodity ILC-flows $K_1$ and $K_2$ is $(r, K)$-minimal, then the multicommodity ILC-flow $K_3$, for which

(48) $$K_3 = K_1 \cup K_2 \text{ and } K_3 \subseteq K,$$

is $(r, K)$-minimal, too.

*Proof.* If $K_1$ is $(r, K)$-minimal then

(49) $$c(r, K) = c(r, K_1)$$

As $K_1 \subseteq K_3$, from Theorem 7 it follows that

(50) $$c(r, K_1) \leq c(r, K_3).$$

From these two relations it is obtained

(51) $$c(r, K_3) \geq c(r, K).$$

As $K_3 \subseteq K$, from Theorem 7 the following inequality is obtained:

(52) $$c(r, K_3) \leq c(r, K).$$

From (51) and (52) the $(r, K)$-minimality of $K_3$ follows.

**Corollary 9.** The union of several multicommodity ILC-flows, from which at least one is $(r,K)$-minimal, is also $(r, K)$-minimal multicommodity ILC-flow.

**Corollary 10.** If $U(r)$ is maximal cutting set, then the $(r, K)$-minimal multicommodity ILC-flow $K_1$ is $K$-minimal.

This result follows from Definition 2, Lemma 3 and Theorems 4 and 7■

Corollary 10 shows that for the determination of a minimal multicommodity ILC-flow there must not use all flows from $K$, but only those which are included in the $K$-minimal multicommodity ILC-flow $K_1 \subseteq K$.

For the determination of the capacity $c(r, K)$ it is also sufficient to use the $(r,K)$-minimal multicommodity ILC-flow $K_1 \subseteq K$.

Upto here, the given results concern the problems for minimality of the multicommodity ILC-flow.

There is a class of problems for feasibility of the multicommodity ILC-flow: given the values $\{v_1^k | k \in K\}$ or $\sum_{k \in K} v_2^k$; to find a realization of the multicommodity ILC-flow, which satisfies those values.

In case of a specific total value $\sum_{k \in K} v_2^k$, according to Lemma 3, an if $r$ is the maximal cutting set, and the following relation holds

(53) $$\sum_{k \in K} v_2^k \geq c(r),$$

then there exists a multicommodity ILC-flow, by which this value can be obtained. At inverse inequality there is no such flow.

At given values $\{v_1^k | k \in K\}$ and inequalities

(54) $$\sum_{k \in K} v_1^k < c(r),$$

there is no multicommodity ILC-flow that satisfies them. According to Theorem 7 the following inequalities are necessary conditions for the flow existence:

(55) $$v_1^k \geq c(r, K_i),\ k_i \in K,\ k_i = K_i.$$

According to Lemma 3 the following inequalities are sufficient:

(56) $$v_1^k \geq w_0^k,\ k \in K,$$

where $w_0^k$ are the values of $w^k$ at $c(r^*)$.

## 4. Conclusion

A class of multicommodity network flow with bounded lower and upper unbounded values of the flow is defined. This flow is called a multicommodity flow with inverse linear constraints or multicommodity ILC-flow. The introduced flow is an extension of the single-commodity ILC-flow.

The capacity of the respective cutting sets is defined on the basis of the inverse linear constraints and a number of relations are proved, characterizing the multicommodity ILC-flow and the capacity of these cutting sets. It is proved that the minimal value of the multicommodity ILC-flow is equal to the capacity of the maximal cutting set (minflow-maxcut theorem).

Some problems for the minimality and feasibility of the multicommodity ILC-flow are considered.

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## Многопродуктовые потоки с обратными линейными ограничениями


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(Р е з ю м е)

Предложен и исследован класс сетевых потоков, названных многопродуктовыми потоками с обратными линейными ограничениями или многопродуктовыми ОЛО-потоками. При этом многопродуктовом потоке значения дуговых потоковых функций, неограниченных сверху и ограниченных снизу множеством линейных неравенств с действительными положительными и отрицательными коэффициентами. Большая часть теоретических результатов однопродуктовых ОЛО-потоков распространяются на исследованном потоке. Использованы рассекающие множества дуг, блокирующие все цепи общего фиктивного источника к общему фиктивному стоку. Введена функция пропускной способности рассекающих множеств и доказан ряд зависимостей для этой функции. Дано доказательство того, что минимальное значение многопродуктового ОЛО-потока равно пропускной способности максимального рассекающего множества (minflow-maxcut теорема). Приведены результаты, относящиеся к задаче допустимости многопродуктового ОЛО-потока.