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Generalizations of Some Criteria for *P*-Matrices and *M*-Matrices

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1. Introduction

It is well known that the set of *P*-matrices includes several important classes of matrices such as *M*-matrices and positive definite matrices. *P*-matrices arize in various theoretical and applied fields, for example in linear complementarity theory [9], in the analysis of the solution set of systems of linear interval equations [6], in the study of convex sets of matrices [5]. The class of *M*-matirces is characterized by a special sign pattern of matrix elements which suggests relations with the theory of nonnegative matrices. From an application point of view, *M*-matrices play an important role in certain economic models [8] and provide a tool for stability analysis of composite dynamical systems [1].

There is a large number of different in form but essentially equivalent conditions that are necessary and sufficient for a given matrix to be a *P*-matrix or an *M*-matrix. Selected lists of such conditions together with the relevant theory are given in [4]. Some generalizations of the *P*-matrix concept related mainly to the linear complementarity problems can be found in [3], [9] and their references. The results in [7] present criteria for the *P*-property of all matrices belonging to an interval matrix set and introduce the notion of interval *P*-matrices.

In this paper, we study the *P*-property and *M*-property of matrices which are elements of compact and convex matrix sets. Our aim is to establish criteria characterizing these properties with respect to all elements of the matrix set. In Theorems 2.1 and 3.1, we have obtained general criteria which are valid for any compact convex set of matrices. Several special cases of compact convex sets found in the literature are also considered. In each case, in addition to the general criterion, we have derived equivalent necessary and sufficient conditions which provide a finite test for the *P*-property and *M*-property of all matrices belonging to the matrix set. The obtained results are based on well known criteria for a single *P*-matrix and *M*-matrix and generalize these criteria to the case of compact and convex matrix sets.

2. Sets of *P*-matrices

Let $\mathbf{R}^n$ and $M_n(\mathbf{R})$ be the normed vector spaces of n-dimensional vectors and $n\times n$ matrices with real elements, respectively. The usual inner product in $\mathbf{R}^n$ will be denoted by $(.,.)$, i.e. $(x, y) = x^T y,\ x, y \in \mathbf{R}^n$. For $x \in \mathbf{R}^n$ ($A \in M_n(\mathbf{R})$) we shall write $x \geq 0$ ($A \geq 0$) if all elements of .$x(A)$ are nonnegative. In this case, $x(A)$ is said to be nonnegative. The positivity of a vector and matrix is defined in a similar way.

A matrix $A \in M_n(\mathbf{R})$ is called a P-matrix if all $k\times k$ principal minors of A are positive for $k = 1, \ldots, n$. The set of $n\times n$ P-matrices will be denoted by $P_n(\mathbf{R})$. It is obvious that $P_n(\mathbf{R}) \subset GL_n(\mathbf{R})$, where $GL_n(\mathbf{R}) \subset M_n(\mathbf{R})$, is the set of nonsingular matrices. The following well known characterizations of a P-matrix [4] will be used in our results.

Given $A \in M_n(\mathbf{R})$, each of the next conditions is necessary and sufficient for $A \in P_n(\mathbf{R})$:

(P1) For every nonzero $x = (x_1, \ldots, x_n)^T \in \mathbf{R}^n$ there is an index $i \in \{1, \ldots, n\}$ such that $x_i(Ax)_i > 0$.

(P2) For every nonzero $x \in \mathbf{R}^n$ there is some nonnegative diagonal matrix $D = D(x) \in M_n(\mathbf{R})$ such that $x^T(D(x)A)\,x > 0$.

(P3) For every nonzero $x \in \mathbf{R}^n$ there is some positive diagonal matrix $D = D(x) \in M_n(\mathbf{R})$ such that $x^T(D(x)A)\,x > 0$.

If $A \in P_n(\mathbf{R})$ then it is also well known that $A^{-1},\ A^T \in P_n(\mathbf{R})$ and $A+D,\ DA,\ AD \in P_n(\mathbf{R})$ for every diagonal matrix D with positive diagonal elements.

The next theorem characterizes the P-property of matrices belonging to a compact convex set of matrices.

Theorem 2.1. Let $K \subset .M_n(\mathbf{R})$ be a compact convex set and let ε be the set of its extreme points. The following conditions are equivalent:

(i) $K \in P_n(\mathbf{R})$

(ii) for every nonzero $x \in \mathbf{R}^n$ there is some nonnegative diagonal matrix $D = D(x) \in M_n(\mathbf{R})$ such that

(1) $$x^T(D(x)A)\,x > 0 \text{ for all } A \in K,$$

(iii) for every nonzero $x \in \mathbf{R}^n$ there is some nonnegative diagonal matrix $D = D(x) \in M_n(\mathbf{R})$ such that

(2) $$x^T(D(x)A)\,x > 0 \text{ for all } A \in \varepsilon.$$

Proof. Since K is compact and convex, it is obvious that (ii) and (iii) are equivalent. Also, (ii) implies (i) by criterion (P2) Hence it remains to show that (i) implies (iii). First, we shall prove this for a polytope of matrices, i. e. $K = \text{convex hull}\{A_1, \ldots, A_m\}$ where $A_i \in M_n(\mathbf{R}),\ i = 1, \ldots, m,$ are given matrices. Let $D \subset M_n(\mathbf{R})$ be defined as $D = \{D = \text{diag}(d_1, \ldots, d_n) \in M_n(\mathbf{R}) : d_i \geq 0, i = 1, \ldots, n\}$. Proceeding by contradiction, assume that i) holds and suppose that for some $x \in \mathbf{R}^n$, $x \neq 0$, there is no matrix $D \in \varepsilon$ satisfying (2). Let $S_1 \subset \mathbf{R}^m$ be the set defined by .

(3) $$S_1 = \{z = (z_1, \ldots, z_n)^T \in \mathbf{R}^m : z_i = x^T DA_i x,\ i = 1, \ldots, m,\ D \in \Delta\},$$

and S_2 be the positive orthant in $\mathbf{R}^m$, i..e:

(4) $$S_2 = \{z = (z_1, \ldots, z_n)^T \in \mathbf{R}^m : z_i > 0,\ i = 1, \ldots, m\}.$$

It is easily seen that S_1 is a closed convex cone in $\mathbf{R}^m$. By our assumption there is no $D \in \Delta$ satisfying (2) and therefore $S_1 \cap S_2 = 0$. Since both S_1 and S_2 are nonempty convex sets, this implies that there exists a hyperplane in $\mathbf{R}^m$ separating S_1 and S_2, i.e. there exists a nonzero vector $a = (a_1, \ldots, a_m)^T \in \mathbf{R}^m$ and real number α such that

(5) $$(a, z) = \alpha_1 z_1 + \ldots + \alpha_m z_m \le \alpha \text{ for all } z \in S_1,$$

(6) $$(a, z) = \alpha_1 z_1 + \ldots + \alpha_m z_m \ge \alpha \text{ for all } z \in S_2 .$$

Inequality (6) implies that $\alpha_i \ge 0$, $i=1, \ldots, m$, and $\alpha_i \le 0$. Indeed, if some $\alpha_i < 0$, then, there always exist $z \in S_2$ for which (6) is violated. Similarly, if $\alpha_i > 0$, then one can find $z \in S_2$ such that (6) is not satisfied. In view of (3), inequality (5) then becomes

(7) $$x^T D(\beta_1 A_1 + \ldots + \beta_m A_m) \le 0 \text{ for all } D \in \Delta,$$

where $\beta_i = \alpha_i / (\alpha_1 + \ldots + \alpha_m)$, $i=1, \ldots, m$. Hence $(\beta_1 A_1 + \ldots + \beta_m A_m) \in K$ and by criterion (P2), $(\beta_1 A_1 + \ldots + \beta_m A_m) \notin P_n(\mathbf{R})$. This contradicts to (i).

Now, let K be any compact convex set of matrices for which condition (i) holds. Then one can find a closed convex polytope K' such that $K \subset K' \subset P_n(\mathbf{R})$. The proof for polytopes shows that for every $x \in \mathbf{R}^n$, $x \ne 0$, there is a nonnegative diagonal matrix $D(x)$ satisfying $x^T(D(x)A)x > 0$ for all $A \in K'$. The inclusion $K \subset K'$ then implies that $x^T(D(x)A)x > 0$ for all $A \in K$, . which completes the proof.

Theorem 2.1 is a generalization of criterion (P2) where condition (iii) of this theorem gives an extreme point characterization for the P-property of every $A \in K$. It is easily seen that both conditions (ii) and (iii) can be analogously formulated using criterion (P3) instead of (P2). In this case, we obtain that $K \subset P_n(\mathbf{R})$ if and only if for every nonzero $x \in \mathbf{R}^n$ there is some positive diagonal matrix $D = D(x) \in M_n(R)$, satisfying (1) or (2). Although (P1), (P2) and (P3) are equivalent for a single matrix $A \in M_n(\mathbf{R})$, the next simple example shows that criterion (P1) can not be generalized in a similar way.

Example 2.2. Let K = convex hull $\{A, B\}$ where

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

It can be easily shown that condition iii) of Theorem 2.1 is satisfied for every $x = (x_1, x_2)^T \in \mathbf{R}^2$, $x \ne 0$. Indeed, if $x_1 = 0$, we take $D(x) = \text{diag}(0, 1)$, so that $x^T(D(x)A)x = x^T(D(x)B)x = x_2^2 > 0$. If $x_1 \ne 0$, then without loss of generality we can consider real vectors $x \in \mathbf{R}^2$ in the form $x = (1, x_2)^T$. In this case, if .$x_2 > 1$ again $D(x) = \text{diag}(0, 1)$ is an appropriate matrix satisfying $x^T(D(x)A)x = x_2^2 > 0$ and $x^T(D(x)B)x = x_2(x_2 - 1) > 0$. Finally, if $x_2 \le 1$, we can choose $D(x) = \text{diag}(1, 1)$ so that $x^T(D(x)A)x = x^T(D(x)B)x = 1 + x_2(x_2 - 1) > 0$. Hence, by Theorem 2.1, every matrix in K is a P-matrix. However, the inclusion $K \subset P_2(\mathbf{R})$ does not imply that for every nonzero $x = (x_1, x_2)^T \in \mathbf{R}^2$, there is some index $i \in \{1, 2\}$ such that $x_i(Ax)_i > 0$ and $x_i(Bx)_i > 0$. In fact, for $x = (1, 1)^T$ we have $x_1(Ax)_1 = 0$, $x_1(Bx)_1 = 1$, and $x_2(Ax)_2 = 1$, $x_2(Bx)_2 = 0$.

In the rest of this section, we study the P-property of several special cases of compact convex sets of matrices. Let J be the matrix whose elements are all equal to 1 and let $\circ$ denotes the Hadamar (entrywise) product of matrices. For any $A, B \in M_n(\mathbf{R})$ the following sets are defined in [5]:

(8) $$K_h = \{C(t) : C(t) = tA + (1 - t)B,\ t \in [0,1]\},$$

(9) $$K_r = \{C(t) : C(t) = TA + (I - T)B,\ T = \text{diag}(t_1, \ldots, t_n),\ t_i \in [0,1]\},$$

(10) $$K_c = \{C(t) : C(t) = AT + B(I - T),\ T = \text{diag}(t_1, \ldots, t_n),\ t_i \in [0,1]\},$$

(11) $$K_i=\{C(t): C(t)=T\circ A+(J-T)\circ B,\ T=(t_{ij}),\ t_{ij}\in[0,1]\},$$

The elements of these sets are matrices $C(t)$, whose entries depend continuously on a scalar or vector parameter t. It is easily seen that K_h, K_r, K_c and K_i are compact and convex, $K_h\subseteq K_r\subseteq K_i$ and $K_h\subseteq K_c\subseteq K_i$. It should be noted that there exist criteria characterizing each of these sets with respect to the nonsingularity of their elements. For example, a well known criterion for nonsingularity of every $A\in K_h$ is the following [2], [5]:

(12) $$K_h\subset GL_n(\mathbf{R}),\ \text{if and only if}\ \sigma(B, A^{-1})\cap(-\infty, 0)=\varnothing,$$

where $\sigma(.)$ denotes the set of eigenvalues (spectrum) of (.). Criteria for nonsingularity of matrices in ... and ... are obtained in [5] as follows:

(13) $$K_r\subset GL_n(\mathbf{R}),\ \text{if and only if}\ BA^{-1}\in P_n(\mathbf{R}),$$

(14) $$K_c\subset GL_n(\mathbf{R}),\ \text{if and only if}\ B^{-1}A\in P_n(\mathbf{R}).$$

It can be easily seen that (11) is an interval matrix set and in this case, various criteria for $K_i\subset GL_n(\mathbf{R})$, are given in [6]. Furthermore, necessary and sufficient conditions for $K_i\subset P_n(\mathbf{R})$ are obtained in [7]. In what follows, we establish criteria for $K_h\subset P_n(\mathbf{R})$, $K_r\subset P_n(\mathbf{R})$ and $K_c\subset P_n(\mathbf{R})$.

Theorem 2.3. Let $AB\in M_n(\mathbf{R})$. The following conditions are equivalent:

(a) $K_h\subset P_n(\mathbf{R})$;

(b) for every nonzero $.x\in R^n$ there is some nonnegative diagonal matrix $D=D(x)\in P_n(\mathbf{R})$ such that $x^T(D(x)A)x>0$ and $x^T(D(x)B)x>0$;

(c) $AB\in P_n(\mathbf{R})$ and

(15) $$\sigma(B_{\alpha\alpha}(A_{\alpha\alpha})^{-1})\cap(-\infty, 0)=\varnothing\ \text{for every}\ \alpha\subseteq\{1, ..., n\},$$

where $A_{\alpha\alpha}$ and $B_{\alpha\alpha}$ are principal submatrices of A and B corresponding to the pair of indices $(\alpha\alpha)$.

Proof. The equivalence of (a) and (b) follows from Theorem 2.1. It will be shown that (a)↔(c). Let $\alpha\subseteq\{1, ..., n\}$. If $K_h\subset P_n(\mathbf{R})$, then obviously $A, B\in P_n(\mathbf{R})$ and $\det C=\det(tA_{\alpha\alpha}+(1-t)B_{\alpha\alpha})>0$ for every $t\in[0, 1]$, which implies (15) by the nonsingularity criterion (12). Conversely, if $A\in P_n(\mathbf{R})$ then for $t=1$ we have

(16) $$\det C_{\alpha\alpha}(t)=\det A_{\alpha\alpha}>0,$$

and from (15), it follows that

(17) $$\det C_{\alpha\alpha}(t)=\det(tA_{\alpha\alpha}+(1-t)B_{\alpha\alpha})\neq 0\ \text{for every}\ t\in[0, 1].$$

Since $\det C_{\alpha\alpha}(t)$ depends continuously on t, conditions (16) and (17) imply that $\det C_{\alpha\alpha}(t)>0$ for every $t\in[0, 1]$, i.e. $C(t)$ is a P-matrix for all values of ...

Theorem 2.4. Let $A, B\in M_n(\mathbf{R})$ and let $\varepsilon_r\subset K_r$ be the set defined as

$$\varepsilon_r=\{C(t): C(t)=TA+(I-T)B,\ T=\operatorname{diag}(t_1, ..., t_n),\ t_i\in[0,1]\}\}.$$

Then the following conditions are equivalent:

(a) $K_r\subset P_n(\mathbf{R})$;

(b) for every nonzero $.x\in\mathbf{R}^n$ there is some nonnegative diagonal matrix

$D = D(x) \in M_n(\mathbf{R})$ such that $x^T(D(x)C(t))\,x > 0$ for all $C(t) \in \varepsilon_r$;

(c) for every nonzero $x \in \mathbf{R}^n$ there is an index $i \in [1, \ldots, n]$ such that $x_i(C(t)x)_i > 0$ for all $C(t) \in \varepsilon_r$;

(d) $AB \in P_n(\mathbf{R})$ and

$$B_{\alpha\alpha}(A_{\alpha\alpha})^{-1} \in P_{|\alpha|}(R) \text{ for every } \alpha \subseteq \{1, \ldots, n\}, \tag{18}$$

where $|\alpha|$ denotes cardinality of α;

(e) $\varepsilon_r \subset P_n(\mathbf{R})$.

Proof. (a)$\leftrightarrow$(b) Follows from Theorem 2.1 by noting that K = convex hull $\{\varepsilon_r\}$.

(a)$\leftrightarrow$(d). As in Theorem 2.3, the proof is based on the fact that (18) is equivalent to $\det C_{\alpha\alpha}(t) = \det(T_{\alpha\alpha}A_{\alpha\alpha} + (I - T_{\alpha\alpha})B_{\alpha\alpha}) \neq 0$ for every $T = \operatorname{diag}(t_1, \ldots, t_n)$, $t_i \in [0,1]$ and that $\det C_{\alpha\alpha}(t)$ depends continuously on t_i, $i = 1, \ldots, n$.

(a)$\leftrightarrow$(e). Since $\varepsilon_r \subset K_r$, it is obvious that (a)$\leftrightarrow$(e). To prove the converse implication, for any $x \in R^n$ we define $C_x \in \varepsilon_r$ as

$$C_x = TA + (I - T)B, \quad T = \operatorname{diag}(t_1, \ldots, t_n) \tag{19}$$

where

$$t_i = \begin{cases} 1, \text{ if } x_i(Ax)_i \le x_i(Bx)_i \\ 0, \text{ if } x_i(Ax)_i > x_i(Bx)_i \end{cases} \quad i = 1, \ldots, n. \tag{20}$$

Then for every $C(t) \in K_r$ and every $i \in \{1, \ldots, n\}$, we have

$$x_i(C(t)x)_i = t_i x_i(Ax)_i + (1 - t_i)x_i(Bx)_i \ge \min\{x_i(Ax)_i, x_i(Bx)_i\} = x_i(C_x x)_i \tag{21}$$

If $\varepsilon_r \subset P_n(\mathbf{R})$ and $x \in \mathbf{R}^n$, $x \neq 0$, then $C_x \in \varepsilon_r$ is a P-matrix and by criterion (P1) there exists some $i \in \{1, \ldots, n\}$, such that $x_i(C_x x)_i > 0$. Inequality (21) now implies

$$x_i(C(t)x)_i \ge x_i(C_x x)_i > 0 \text{ for all } C(t) \in K_r. \tag{22}$$

and hence, we obtain that $K_r \subset P_n(R)$.

(c)$\leftrightarrow$(e). If condition (c) holds, then $\varepsilon_r \subset P_n(\mathbf{R})$ by criterion (P1). The converse implication follows from inequality (22) where $C_x \in \varepsilon_r$ is given by (19) and (20). This completes the proof.

In a similar way, we obtain the following necessary and sufficient conditions for $K_c \subset P_n(\mathbf{R})$.

Theorem 2.5. Let $AB \in M_n(\mathbf{R})$ and let $\varepsilon_c \subset K_c$ be the set defined as

$\varepsilon_c = \{C(t) : C(t) = AT + B(I - T),\ T = \operatorname{diag}(t_1, \ldots, t_n),\ t_i \in \{0,1\}\}$.

Then the following conditions are equivalent:

(a) $K_c \subset P_n(\mathbf{R})$;

(b) for every nonzero $x \in \mathbf{R}^n$ there is some nonnegative diagonal matrix $D = D(x) \in M_n(\mathbf{R})$ such that $x^T(D(x)C(t))\,x > 0$ for all $C(t) \in \varepsilon_r$;

(c) for every nonzero $x \in \mathbf{R}^n$ there is an index $i \in [1, \ldots, n]$ such that $x_i(C(t)x)_i > 0$ for all $C(t) \in \varepsilon_c$;

(d) $AB \in P_n(\mathbf{R})$ and

$$(B_{\alpha\alpha})^{-1}A_{\alpha\alpha} \in P_{|\alpha|}(\mathbf{R}) \text{ for every } \alpha \subseteq \{1, \ldots, n\},$$

where $|\alpha|$ denotes cardinality of α;

(e) $\varepsilon_c \subset P_n(\mathbf{R})$.

Note that ε_r and ε_c in Theorems 2.4 and 2.5 are the sets of extreme points of K_r and K_c respectively. Conditions (e) in both theorems show that $K_r \subset P_n(\mathbf{R})$ (respectively, $K_c \subset P_n(\mathbf{R})$) if and only if the extreme matrices of K_r (respectively, K_c) are P-matrices. Since the cardinality of ε_r (ε_c) is 2^n, these conditions provide a finite test for the P-property of every matrix in K_r or K_c. Condition (c) in Theorem 2.3 and conditions (d) in Theorems 2.4 and 2.5 are essentially based on the nonsingularity criteria (12), (13) and (14), respectively. These conditions also lead to a finite characterization of the P-property of matrices in K_h, K_r and K_c. Finally, conditions (c) in Theorems 2.4 and 2.5 show that, for a certain type of convex matrix sets, criterion (P1) still can be generalized analogously as (P2) and (P3). A similar conclusion follows from [7] where the P-property of interval matrices is characterized in terms of condition (P1).

Example 2.6. Let K_h, K_r and K_c be given by (8), (9) and (10), respectively, where the values of A and B are taken from [5] as follows

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

For every $\alpha \subseteq \{1, 2, 3\}$, it can be easily checked that $\sigma(B_{\alpha\alpha})(A_{\alpha\alpha})^{-1}) \cap (-\infty, 0] = \emptyset$, and hence by condition (c) of Theorem 2.3 $K_h \subset P_n(\mathbf{R})$. Also, every extreme matrix of K_c is a P-matrix and by Theorem 2.4 $K_r \subset P_3(\mathbf{R})$. However, for $T = \mathrm{diag}\,(1, 0, 0)$, we have

$$C = AT + B(I - T) = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

which is an extreme point of K_c and obviously $\det C = 0$. Thus, $K_r \not\subset P_3(\mathbf{R})$ and $K_c \not\subset GL_3(\mathbf{R})$.

3. Sets of M-matrices

Let $Z_n(\mathbf{R}) \subset P_n(\mathbf{R})$ be the set defined by

$$Z_n(\mathbf{R}) = \{A = (a_{ij}) \in M_n(\mathbf{R}) : a_{ij} \le 0 \text{ if } i \ne j,\ i, j = 1, \ldots, n\}.$$

The elements of $Z_n(\mathbf{R})$ are related with the nonnegative matrices by the following representation: $A \in Z_n(\mathbf{R})$ if and only if $A = \alpha I - P$ for some $\alpha \in R$ and some $P \in M_n(\mathbf{R})$ with $P \ge 0$.

A matrix $A \in M_n(\mathbf{R})$ is called an M-matrix if $A \in Z_n(\mathbf{R})$ and the eigenvalues of A all have positive real parts. The set of $n \times n$ M-matrices will be denoted by $M_n(\mathbf{R})$. A well known criterion [4] states that $A \in M_n(\mathbf{R})$ if and only if $A \in Z_n(\mathbf{R})$ and all $k \times k$ principal minors of A are positive for $k = 1, \ldots, n$. Thus, we have

(23) $$M_n(\mathbf{R}) = Z_n(\mathbf{R}) \cap P_n(\mathbf{R}).$$

In view of the special structure of M-matrices ($M_n(\mathbf{R}) \subset Z_n(\mathbf{R})$), criterion (P2) from the previous section can be specialized to the following necessary and sufficient condition.

$A \in Z_n(\mathbf{R})$ is an M-matrix if and only if for every nonzero $x \in R^n$ with $x \ge 0$ there is some nonnegative diagonal matrix $D = D(x) \in M_n(\mathbf{R})$ such that

(24) $$x^T(D(x)A)\,x > 0.$$

If $A \in M_n(\mathbf{R})$ then the inclusion $M_n(\mathbf{R}) \subset P_n(\mathbf{R})$ implies (24) by (P2). To show that (24) is also sufficient, let $x = (x_1, \ldots, x_n)^T \in \mathbf{R}^n$ and let $|x|$ denote $|x| = (|x_1|, \ldots, |x_n|)^T$. Then for every $A \in Z_n(\mathbf{R})$ and every nonnegative diagonal matrix D, we have

(25) $$x^T(D(x)A = x^T(\alpha D - DP)x \geq |x|^T(\alpha D - DP)\,|x| = |x|^T(DA)\,|x|.$$

In (25), A is represented as $A = \alpha I - P$ where $\alpha \in R$ and $P \geq 0$ and the inequality follows from the fact that $DP \geq 0$ which implies $x^T(DP)\,x \leq |x|^T(DP)\,|x|$. If condition (24) holds, then for every nonzero x there is a nonnegative diagonal matrix $D(x)$ satisfying $|x|^T(D(x)A)\,|x| > 0$. From (25) it also follows that $x^T(D(x)A)\,x > 0$ and hence $A \in P_n(\mathbf{R})$ by criterion (P2). Since $A \in Z_n(\mathbf{R})$ and $A \in P_n(\mathbf{R})$, we obtain that A is an M-matrix.

Now, let $K \subset M_n(\mathbf{R})$, be a compact convex set of matrices. Using (23), one can obtain necessary and sufficient conditions for $K \subset .M_n(\mathbf{R})$,. similarly as in Theorem 2.1 with the additional requirement that $K_r \subset Z_n(\mathbf{R})$,. On the other hand, there is a close relation between nonsingularity and M-property which also can be used to characterize the inclusion $K \subset M_n(\mathbf{R})$. As a result, we obtain the following theorem.

Theorem 3.1. Let $K \subset Z_n(\mathbf{R})$, be a compact convex set and let ε be the set of its extreme points. The following conditions are equivalent:

(a) $K \subset M_n(\mathbf{R})$;

(b) for every nonzero $.x \in \mathbf{R}^n$ with $.x \geq 0$, there is some nonnegative diagonal matrix $D = D(x) \in M_n(\mathbf{R})$ such that

(26) $$x^T(D(x)A)\,x > 0. \text{ for all } A \in \varepsilon;$$

(c) there is some $A_0 \in K$ which is an M-matrix and

(27) $$K \subset GL_n(\mathbf{R}).$$

Proof. (a)$\leftrightarrow$(b). Since $M_n(\mathbf{R}) \subset P_n(\mathbf{R})$, (a) implies (b) by condition (iii) of Theorem 2.1. The converse implication follows from convexity of K and (24).

(a)$\leftrightarrow$(c). Obviously, (a)$\rightarrow$(c). It will be shown that (c)$\rightarrow$(a). For any $A \in Z_n(\mathbf{R})$, let $\tau(A)$ be defined as $\tau(A) = \min\{\mathrm{Re}\,\lambda, \lambda \in \sigma(A)\}$. It is well known [4] that $\tau(A) \in \sigma(A)$, i.e. $\tau(A)$ is a real eigenvalue of A with the property that $\tau(A) \leq \mathrm{Re}\lambda, \lambda \in \sigma(A)$. Now, if $A_0 \in K$ is an M-matrix, we have $\tau(A_0) > 0$. Since K is convex and $\tau(A)$ depends continuously on A, condition (27) implies that $\mathrm{Re}\,\lambda \geq \tau(A) > 0$, $\lambda \in \sigma(A)$ for all $A \in K$, i.e. $K \subset M_n(\mathbf{R})$..

Similarly as in Theorems 2.3, 2.4 and 2.5, one can obtain various criteria characterizing matrix setst K_h, K_r and K_c with respect to the M-property of their elements. We shall state here only the simple necessary and sufficient conditions following from condition (c) of Theorem 3.1 and the nonsingularity criteria (12), (13) and (14). Note that each of t K_h, K_r and K_c is a subset of $Z_n(\mathbf{R})$ if and only if $A, B \in Z_n(\mathbf{R})$.

Concerning (8), a criterion given in [4] states that $K_h \subset M_n(\mathbf{R})$ if and only if $A, B \in M_n(\mathbf{R})$ and $\sigma(BA)^{-1}) \cap (-\infty, 0] = \emptyset$. Obviously, this result follows immediately from Theorem 3.1 and (12). The following corollary is also a simple consequence of this theorem and conditions (13) and (14).

Corollary 3.2. Let $A, B \in M_n(\mathbf{R})$. Then $K_h \subset M_n(\mathbf{R})$ if and only if $A, B \in Z_n(\mathbf{R})$. and $B A^{-1} \in P_n(\mathbf{R})$. Similarly, $K_c \subset M_n(\mathbf{R})$ if and only if $A, B \in M_n(\mathbf{R})$ and $B^{-1}A \in P_n(\mathbf{R})$.

4. Conclusion

In this paper, we have obtained necessary and sufficient conditions characterizing the P-property and M-property of matrices belonging to a certain type of convex matrix sets. The most general results are presented in Theorems 2.1 and 3.1 which are valid for any compact convex set of matrices. The obtained criteria are formulated in terms of the extreme points of the set and generalize the well known criteria for a single *P*-matrix and *M*-matrix [4]. In comparison with the results on interval P-matrices [7], Theorem 2.1 further extends the class of matrix sets which are characterized by the P-property of their elements. Several special cases of compact convex sets of matrices are considered in Theorems 2.3, 2.4 and 2.5, respectively. These sets have been defined and studied in [5] with respect to the nonsingularity of their elements. Here, we have derived criteria which enable to establish the P-property and M-property of all elements in the correspondig set. Some of the obtained criteria are based on the results of [5] and show that there is a close relation between nonsingularity and P-property of matrices belonging to convex matrix sets.

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Обобщение некоторых критериев для *P*–матриц и *M*–матриц

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(Резюме)

Рассматриваются два класса специальных матриц – *P*–матриц и *M*–матриц, которые играют важную роль в области приложной линейной алгебры, интервального анализа и исследования динамического поведения больших систем управления. Дискутируются некоторые критерии, характеризирующие заданную квадратную матрицу, такие как *P*–матрицу или *M*–матрицу. Основные результаты представлены в пяти теоремах.