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On a Network Flow with One Linear Equality*

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1. Introduction

The network flow theory has passed through different stages of development since its creation in the 50ies. This has lead to new classes of flows, that differ mainly in the way of defining the capacity of the network arcs. The main results in the already classical network flows are connected with the names of Ford and Fulkerson [1, 2, 3]. The capacity in these flows is defined separately for each network arc.

Some types, in which additional linear inequalities of the flow on subsets of arcs are used together with the arc capacities, have been considered in [4, 5].

A more general approach is suggested in [6, 7], where the arc capacities are replaced by linear constraints of the flow on separate subsets of arcs. This flow is called a linear flow.

The present paper discusses a class of network flows, in which all the arc capacities are replaced by one linear equality with a non-negative right side. In this case the arc flow functions are not limited by the capacities, but they can alter within certain limits with respect to a common linear equality. Further on this flow will be called a network flow with one linear equality (OLE) or OLE-flow in brief.

2. Definition of a network flow with one linear equality

Let a graph $G(N,U)$ be given with a set of nodes N and a set of arcs U. The set M contains the indices of all simple oriented paths [1, 2] from the source s towards the sink t, such, that there are no knots in them and every arc is included only once in each path. Since simple oriented paths will be only used further on, they will be called just paths.

The set of all the arcs with an index $\mu \in M$ will be denoted as $U(\mu)$. We will assume that $G(N,U)$ has no oriented cycles and for it

(1) $$U = \bigcup_{\mu \in M} U(\mu).$$

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Let a network flow be given on the graph $G(N,X)$ with the help of the following constraints: for each $x \in N$

(2) $$f(x, N) - f(N, x) = \begin{cases} v \text{ if } x = s \\ 0 \text{ if } x \neq s, t \\ -v \text{ if } x = t, \end{cases}$$

(3) $$\sum_{(x,y) \in U} b(x, y)\, f(x, y) = C,$$

(4) $$f(x, y) \geq 0;\ (x, y) \in U,$$

where C is a rational non-negative number; s and t – a source and a sink of the network flow; v and f – a flow and an arc flow function.

(5) $$f(x, N) = \sum_{y \in \Gamma^1(x)} f(x, y); \qquad f(N, x) = \sum_{y \in \Gamma^{-1}(x)} f(y, x);$$

(6) $$b(x, y) \in R';$$

R' – a set of rational non-negative numbers;

$\Gamma^1(x)$ and $\Gamma^{-1}(x)$ – an image and an inverse image of x into N.

It is assumed that for the graph $G(N,U)$

(7) $$(N, s) = (t, N) = \Lambda,$$

where Λ is an empty set.

The network flow defined with the help of relations (1)–(3) will be called **a flow with one linear equality or OLE-flow**.

Let $v(\mu)$ denotes this part of the flow v, corresponding to the path μ. Having in mind condition (7) and Theorem 2.2 from chapter 2 in [1], the following transition from a network flow v in the type of arcs-nodes towards a flow $v(h)$ in arcs-paths type can be done,

(8) $$v(h) = \sum_{\mu \in M} v(\mu) = v.$$

The denotation v only will be used further on.

The sum of all the coefficients $\{b(x,y)\}$, corresponding to the path $\mu \in M$ will be denoted as $B(\mu)$, i.e.,

(9) $$B(\mu) = \sum_{(x,y) \in U(\mu)} b(x,y).$$

The coefficients $\{\alpha_\mu\}$ are determined as follows: for each $\mu \in M$

(10) $$\alpha_\mu = \begin{cases} v(\mu)/v \text{ if } v>0 \\ 0 \text{ if } v=0. \end{cases}$$

It follows from the non-negativeness of the flows and relations (8) and (10), that:

(11) $$\sum_{\mu \in M} \alpha_\mu = \begin{cases} 1 \text{ if } v>0 \\ 0 \text{ if } v=0, \end{cases}$$

(12) $$0 \leq \alpha_\mu \leq 1 \text{ for all } \mu \in M$$

Let the paths number in M be equal to m, i. e., $|M| = m$. In case the indices of the paths are ordered in the respective way, the values $\{\alpha_\mu\}$ can be considered as components of the following vector:

$$\alpha=(\alpha_1, \alpha_2, \dots, \alpha_m).$$

Definition 1. The value

$$B_\alpha = \sum_{\mu\in M} \alpha_\mu B(\mu) \tag{13}$$

will be called an α-factor.

The value $B(\mu)$ depends on the coefficients $\{b(x, y) / (x, y)\in U(\mu)\}$ only according to (9) and is an apriori known value. According to (10) and (11) the α-factor depends on the components $\{\alpha_\mu\}$ only, i.e. on the corresponding flow realization.

Definition 2. Each value of the vector α, satisfying conditions (8) and (10) will be called an α- realization.

It follows from the relations in (8) and (10) that if the equalities (3) are not accounted, each α-realization can be assigned some sets of flow values.

The following two statements follow directly from relations (8)–(13).

Statement 1. Only one α-factor corresponds to any α-realization.

Statement 2. There exists a biunique correspondence between each α-realization and the respective flow realization $\{v(\mu)/\mu\in M\}$ for a given value of v.

3. Conditions for the existence of a network flow with one linear equality

The following relations have a significant influence on the OLE-flow.

Lemma 1. *It can be written for the OLE-flow, described by (2)–(4)*

$$\sum_{(x, y)\in U} b(x, y) f(x, y) = \sum_{\mu\in M} v(\mu) B(\mu); \tag{14}$$

P r o o f. Having in mind [1], the following coefficients are introduced: for each $\mu \in M$

$$\alpha_\mu(x, y) = \begin{cases} 1 \text{ if } (x, y)\in U(\mu) \\ 0 \text{ otherwise.} \end{cases} \tag{15}$$

When passing from a flow in the form of arcs-nodes towards a flow in the form of arcs-paths it can be written, [1]: for each $(x, y)\in U$

$$f(x, y) = \sum_{\mu \in M} \alpha_\mu(x, y)\, v(\mu). \tag{16}$$

Then on the basis of (8), (15) and (16) the left part of equality (14) can be written as the following chain of equalities:

$$\sum_{(x, y)\in U} b(x, y) f(x, y) = \sum_{(x, y)\in U} b(x, y) \sum_{\mu\in M} a_\mu(x, y)\, v(\mu)= \tag{17}$$

$$=\sum_{\mu\in M} v(\mu) \sum_{(x, y)\in U} a_\mu(x, y) b(x, y) =\sum_{\mu\in M} v(\mu) \sum_{(x, y)\in U(\mu)} b(x, y).$$

From (9) and (17), (14) follows ▫

Consequence 2. *There exists a relation*

$$\sum_{(x, y)\in U} b(x, y) f(x, y) = vB_\alpha \tag{18}$$

This result follows directly from relations (8)–(14) ∎

The relations below given are connected with the conditions of existence and behaviour of an OLE-flow (2)–(4), as well as with the capacities, limiting this flow.

Lemma 3. *If*

(19) $$C>0 \text{ and } B(\mu)\leq 0 \text{ for all } \mu\in M,$$

there does not exist any OLE-flow, satisfying conditions (2) upto (4).

P r o o f. Let

(20) $$B(\mu)\leq 0 \text{ for all } \mu\in M.$$

Then for each flow realization $\{v(\mu)/\mu\in M\}$ it can be written

(21) $$\sum_{\mu\in M} v(\mu)B(\mu)\leq 0.$$

The comparison of relations (3), (14) and (21) leads to the inequality $C\leq 0$, which contradicts to the first one of inequalities (19) and proves the impossibility of an OLE-flow existence under conditions (19)∎

Lemma 4. *If*

(22) $$C>0,$$

the necessary and sufficient condition for the existence of a positive OLE-flow from (2) upto (4) is the presence of at least one path $\mu'\in M$, *for which the following is true:*

(23) $$B(\mu')>0.$$

P r o o f. Sufficiency. Let a path $\mu'\in M$ exists, which satisfies (23). Then a flow $v(\mu')=v$ can be set, such that,

(24) $$v(\mu')=\frac{C}{B(\mu')};$$

(25) $$v=\sum_{\mu\in M} v(\mu)=v(\mu').$$

From (14) and from (22) upto (25), it follows that all the constraints (2)–(4) are fulfilled for this flow and hence it is an OLE-flow.

Necessity. Let a non-zero flow $v>0$ exists. It follows from assumption (22) and relations (3) and (14) that

(26) $$\sum_{\mu\in M} v(\mu)\,B(\mu)=C>0.$$

This means that there is at least one path $\mu'\in M$, fulfilling (23)∎

Lemma 5. *If*

(27) $$C=0 \text{ и } B(\mu)\neq 0 \text{ for all } \mu\in M,$$

the necessary and sufficient condition for the existence of a non-zero OLE- flow is the presence of at least two paths $\mu'\in M$ and $\mu''\in M$, for which

(28) $$B(\mu')>0 \text{ and } B(\mu'')<0;$$

P r o o f. Sufficiency. The existence of two paths $\mu'\in M$ and $\mu''\in M$, satisfying condition (28), is assumed.

An arbitrary positive number $P\in R'$ is chosen, which is assigned to the following flow functions:

(29) $$f(x, y) = P \text{ for all } (x, y) \in U(\mu').$$

The arc flow function of the second path μ'' is assigned the following values

(30) $$f(x, y) = \frac{B(\mu')}{B(\mu'')} P \text{ for all } (x, y) \in U(\mu'').$$

It is obvious that

(31) $$v(\mu'') = -\frac{B(\mu')}{B(\mu'')} P.$$

For the rest of the arcs in U it is accepted

(32) $$f(x, y) = 0, \text{ if } (x, y) \in U \setminus (U(\mu') \cup U(\mu'')).$$

The comparison of the previous four relations with conditions (2)–(4) shows that a non-zero OLE-flow has been constructed for which

(33) $$v = v(\mu') + v(\mu'') = P\left(1 - \frac{B(\mu')}{B(\mu'')}\right);$$

(34) $$v(\mu')B(\mu') + v(\mu'')B(\mu'') = 0.$$

Necessity: Let a non-zero OLE-flow be given. Then from condition (27) and relations (3) and (14) it follows:

(35) $$\sum_{\mu \in M} v(\mu)B(\mu) = C = 0.$$

In case it is assumed that $B(\mu) < 0$ for all $\mu \in M$, it is impossible to satisfy condition (35). Hence, there exists at least one path $\mu' \in M$, for which (23) is fulfilled. But then in order to satisfy (35), at least one path $\mu'' \in M$, is required, for which $B(\mu'') < 0$.

Lemma 6. *If*

(36) $$C = 0$$

and if there is at least one path $\mu \in M$, *for which*

(37) $$B(\mu) = 0,$$

then there exists an OLE-flow.

Proof. The network arcs are assigned the following flow functions

(38) $$f(x, y) = \begin{cases} v(\mu) \geq 0 \text{ if } (x, y) \in U(\mu) \\ 0 \text{ in the rest of the cases.} \end{cases}$$

From (14) and from (36) upto (38) it follows that these functions satisfy all the conditions (2)–(4), and v can accept any non-negative value.

4. Capacity of a network flow with one linear equality

The following denotations are introduced:

(39) $$B^1 = \min_{\mu \in M} B(\mu), \quad B^2 = \max_{\mu \in M} B(\mu);$$

(40) $$C^1 = \begin{cases} C/B^1 \text{ if } B^1 > 0 \\ +\infty \text{ in the rest of the cases;} \end{cases}$$

(41) $$C^2 = \begin{cases} C/B^2 \text{ if } C>0 \\ 0 \text{ in the rest of the cases.} \end{cases}$$

Theorem 7. *If there exists an OLE-flow v, the following is valid for it:*

(42) $$C^2 \le v \le C^1.$$

Proof 1. It is assumed that

(43) $$v > C^1.$$

1.1. Let

(44) $$B^1 > 0.$$

Then it follows from (39) that a path $\mu' \in M$ can be found for which

(45) $$B(\mu') = B^1 = \min_{\mu \in M} B(\mu).$$

It follows from (3), (10) and (14) that

(46) $$v \sum_{\mu \in M} \alpha_\mu B(\mu) = C.$$

If v from (46) is placed in the left side of (43), and in the right side – the value C^1 from (40), corresponding to the case (44), the following inequality will be true:

(47) $$\frac{C}{\sum_{\mu \in M} \alpha_\mu B(\mu)} > \frac{C}{B^1},$$

i.e. the relation

(48) $$\sum_{\mu \in M} \alpha_\mu B(\mu) < B^1,$$

which contradicts to the definitions from (10) upto (12) and (39) and verifies the impossibility of assumptions (43) and (44).

1.2. Let

(49) $$B^1 \le 0.$$

In this case, according to (40), the value C^1 converges towards $+\infty$, which makes (43) impossible.

This makes assumption (43) invalid in both the cases – (44) and (49).

2. It is assumed that

(50) $$v < C^2.$$

2.1. Let

(51) $$C > 0.$$

It follows from (39) that a path $\mu'' \in M$ will be found for which

(52) $$B(\mu'') = B^2 = \max B(\mu).$$

$\mu \in M$

In case v from (46) is put in the left side of (50), and in the right side – the value C^2 from (41), corresponding to the case (51), this will lead to the relation

$$\frac{C}{\sum_{\mu \in M} \alpha_{\mu} B(\mu)} < \frac{C}{B^2}. \tag{53}$$

The following inequality follows from that:

$$\sum_{\mu \in M} \alpha_{\mu} B(\mu) > B^2,$$

which contradicts to definitions (10)–(12) and (39) and verifies the impossibility of (50) in case (51).

2.2. Let

$$C = 0. \tag{54}$$

In this case, according to (41)

$$C^2 = 0. \tag{55}$$

From the non-negativeness of the flow v and (55), the impossibilty of inequality (50) in case (54) follows.

Hence, the assumption (50) is not true in both cases – (51) and (54).

The invalidity of both inequalities (43) and (50) proves (42).

The following six relations follow directly from relations (39) upto (42) and Lemmas 3 and 4.

Consequence 8. *If* $B^1 > 0$ and $C > 0$, *then* $0 < C/B^1 \leq v \leq C/B^2 < +\infty$.

Consequence 9. *If* $B^1 \leq 0$ and $C = 0$, *then* $0 \leq v \leq +\infty$.

Consequence 10. *If* $B^1 > 0$ and $C = 0$, *then* $C^2 = v = C^1 = 0$.

Consequence 11. *If* $B^1 \leq 0$ and $C > 0$, *then* $0 \leq C/B^2 \leq v \leq +\infty$.

Consequence 12. *If* $B^1 = B^2$ and $C > 0$, *then* $0 < C^2 = v = C^1 < +\infty$.

Consequence 13. *If* $B^1 = B^2$ and $C = 0$, *then* $C^2 = v = C^1 = 0$.

These results enable the introducing of OLE-flow capacity denotation.

Definition 3. The values C^1 and C^2 from (40) *and* (41) *will be called upper and lower limit of the capacity of the OLE-flow* (2)–(4) *respectively.*

The following two statements follow directly from inequalities (42) of Theorem 7.

Statement 3. The maximal OLE-flow v_{max} is equal to the upper limit C^1 of the capacity.

Statement 4. The minimal OLE-flow v_{min} is equal to the lower limit C^2 of the capacity.

In case the OLE-flow (2)–(4) exists, its maximal and minimal value can be determined in general with the help of the following linear programming problems respectively: v_{max} and v_{min} subject to constraints (2) upto (4).

5. Results discussion

There are some important differences between the classical network flows and the OLE-flow discussed.

While in the classical flows the capacity is defined as the exact upper and/or lower limit of the flow function on each arc, in OLE-flow it is not fixed on each arc, but a linear equality is set for all the arcs with a given non-negative right side. At that it is not apriori known what the values of the arc flow functions are at maximal and minimal values of the network flow.

There always exists at least one flow realization for the classical network flows, corresponding to the arc flow functions with non-zero values. As it follows from Lemma 3, for some values of the coefficients $\{b(x, y)\}$, the OLE-flow may not exist.

Statements 3 and 4 for the OLE-flow can be regarded as an analogue for this flow of the famous theorem of L.R. Ford and D. R. Fulkerson about the maximal flow and minimal cutting section [1, 2, 3], which has an important role in classical network flows.

The results in the present paper have been obtained for an OLE-flow in the form of arcs-paths without the use of cutting sections. It would be useful to investigate an OLE-flow in the arcs-nodes form applying the approach of a network section. The development of specific efficient network algorithms is necessary in order to find the maximal and minimal OLE-flow, as well as an OLE-flow of minimal estimate.

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Сетевой поток с линейным равенством

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(Резюме)

Рассматривается класс потоков в сетях, в котором не задаются пропускные способности отдельных дуг, а дуговые потоки ограничиваются одним общим линейным равенством с неотрицательной правой частью. Даны некоторые условия существования этого сетевого потока с одним линейным равенством. Вводятся понятия нижней и верхней границами пропускной способности этого потока и приводятся результаты, связанные с поведением потоковых функций.