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An Interactive Reference Direction Algorithm of the Convex Nonlinear Integer Multiobjective Programming

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1. Introduction

The nonlinear problems with continuous variables and the convex integer (linear and nonlinear) programming problems are NP-hard [2, 3]. The exact algorithms to solve these problems have exponential computational complexity. The integer problems are further characterized by the fact that, at times, finding a feasible solution is as difficult as finding an optimal solution.

When developing an interactive algorithm to solve multiple objective nonlinear and integer (linear and nonlinear) programming problems, it is imperative to take into consideration the time required to solve the scalarizing problems. If it takes too long to solve these problems, the dialogue with the DM, even if very convenient, may not take place. This may happen if the DM is unwilling to wait a long time for the solution to the scalarizing problem.

An approach to overcome some of the difficulties due to the computational complexities associated with solving the multiple objective linear integer programming problems has been proposed [5, 7]. It is among one of the most innovative among the interactive algorithms designed to solve multiple objective linear integer programming problems [5, 7, 8, 9]. The main feature of this approach is that the solutions to the single objective linear problems with continuous variables are presented to the DM for evaluation. These problems are easy to solve. They are used under the assumption that the objective function values for the scalarizing problem with continuous variables differ comparatively little from the solutions with integer variables. It is further supposed that the DM prefers to deal in the objective function rather than the variable space. The advantage of these interactive algorithms is that the computational time expended to obtain a new solution for evaluation by the DM is improved without deteriorating the quality of the dialogue with the DM.

Unfortunately this approach loses some of its advantages when multiple objective convex nonlinear integer problems must be solved. The scalarizing convex nonlinear integer problems are difficult to solve. The corresponding single objective convex nonlinear problems with continuous variables are also difficult to solve [10]. Therefore it is unattractive to use such problems during initial phase of the procedure.

In this paper, our objective is to propose an interactive algorithm that overcomes some of the computational complexities associated with solving multiple objective convex nonlinear integer problems. It belongs to the class of reference direction algorithms. The DM sets his preferences as aspiration levels of the objective functions. The modified aspiration point in the objective functions space and the solution found at the previous iteration define the reference direction. The modified aspiration point is obtained from the aspiration point set by the DM by replacing the aspiration levels of the objective functions, which the DM agrees to worsen, by their values in the solution found at the previous iteration. Based on the reference direction, a scalarizing problem is formulated.

2. The problem statement

The proposed algorithm is designed to solve the following multiple objective convex nonlinear integer programming problems:

(1) $$\max \{ f_k(x) \mid k \in K\}$$

subject to:

(2) $$g_i(x) \le 0, \ \ i \in M,$$

(3) $$0 \le x_j \le d_j, \ j \in J,$$

(4) $$x_j - \text{integer}, \quad j \in J,$$

where $f_k(x)$, $k \in K = \{1, 2, ..., p\}$, are concave functions; $g_i(x)$, $i \in M = \{1, 2, ..., m\}$ are convex functions; and $J = \{1, 2, ..., n\}$. Without any loss of generality, the symbol "max" implies that each objective function has to be maximized. The constraints (2)–(4) define the feasible set X.

We give a few definitions primarily to improve the clarity of the text:

Definition 1. The solution x is called efficient if there does not exist another solution $\bar{x}$, such that the inequalities

$$f_k(\bar{x}) \ge f_k(x) \text{ for each } k \in K$$

and

$$f_k(\bar{x}) \ge f_k(x) \text{ for at least one } k \in K$$

hold.

Definition 2. The solution x is called weak efficient if there does not exist another solution $\bar{x}$, such that

$$f_k(\bar{x}) > f_k(x) \text{ for each } k \in K.$$

Definition 3. The p-dimensional vector $f(x)$ with components $f_k(x)$, $k \in K$, is called (weak) nondominated, if x is an (weak) efficient solution.

Definition 4. The near (weak) nondominated solution is a feasible solution in the objective functions space, located near the (weak) nondominated solutions.

Definition 5. An aspiration point in the objective functions space is a point whose components values are defined by the aspiration levels of the objective functions set by the DM.

Definition 6. A modified aspiration point is obtained from an aspiration point, set by the DM, by replacing the aspiration levels of the objective functions, which the DM agrees to worsen, by their values in the last solution found.

Definition 7. The reference direction is defined by the difference between the modified aspiration point and the solution found at the previous iteration.

Let f_k and $\bar{f}_k$ denote the value of the k-th, $k \in K$, objective function found at the last iteration and its desired value (aspiration level) defined by the DM at the current iteration, respectively. Further, let

$$K1 = \{k \in K \mid \bar{f}_k > f_k\},\ K2 = \{k \in K \mid \bar{f}_k < f_k\},$$

$$K3 = \{k \in K \mid \bar{f}_k = f_k\}, \quad K = K1 \cup K2 \cup K3.$$

The set $K1$ contains indices of those objective functions whose values the DM wishes to improve and the set $K2$ contains the indices of those objective functions for which the DM agrees to be worse. The set $K3$ contains the indices of the objective functions whose values the DM is unwilling to deteriorate.

The aspiration levels $\bar{f}_k$, $k \in K$, of the objective functions, set by the DM, define the aspiration point (reference point). Let us determine the components $\tilde{f}_k$, $k \in K$, of a modified aspiration point (modified reference point) as follows:

$$\tilde{f}_k = \begin{cases} \bar{f}_k, & k \in K1 \cup K3 \\ f_k, & k \in K. \end{cases}$$

The modified aspiration point differs from the DM aspiration point in this, that the aspiration levels $\bar{f}_k$, $k \in K_2$, are replaced by the values of the objective functions in the last solution found.

On the basis of the modified aspiration points and the last solution found the following single objective problem is proposed in order to obtain a (weak) nondominated solution:

Minimize

(5) $$S(x) = \max \left[\max_{k \in K1} (\bar{f}_k - f_k(x))/(\bar{f}_k - f_k), \max_{k \in K2} (f_k - f_k(x))/(f_k - \bar{f}_k) \right]$$

subject to

(6) $$f_k(x) \geq \bar{f}_k,\ k \in K2 \cup K3,$$

(7) $$x \in X.$$

It should be noted that problem (5)–(7) has a feasible solution if the feasible set X is non-empty, and has an optimal solution if the objective functions $f_k(x)$, $k \in K$, are finite over X.

The basic feature of the scalarizing problem (5)–(7) is the minimization of the maximal standardized deviation of the solution to be found, $f_k(x)$, $k \in K$, and the modified aspiration point $\tilde{f}_k$, $k \in K$, in the objective functions space satisfying constraints (6)–(7).

Theorem. The optimal solution for problem (5)–(7) is a weak efficient solution for problem (1)–(4).

Proof. Let $K1 \neq \emptyset$ and let x^* be the optimal solution of problem (5)–(7). Then the following inequality holds:

(8) $$S(x^*) \leq S(x) \text{ for each } x \in X \text{ and } f_k(x^*) \geq \bar{f}_k,\ k \in K2 \cup K3.$$

Let us assume that x^* is not a weak efficient solution for problem (1)–(4). There exists a point $x' \in X$, such that

(9) $$f_k(x^*) < f_k(x') \text{ for } k \in K \text{ and } f_k(x^*) \geq \bar{f}_k,\ k \in K2 \cup K3.$$

After transforming the objective function $S(x)$ of problem (5)–(7), using inequality (9), the following relation:

(10) $$\begin{aligned} S(x') &= \max[\max_{k\in K1}(\bar{f}_k - f_k(x'))/(\bar{f}_k - f_k), \max_{k\in K2}(f_k - f_k(x'))/(f_k - \bar{f}_k)] \\ &= \max[\max_{k\in K1}(\bar{f}_k - f_k(x^*)) + (f_k(x^*) - f_k(x')))/(\bar{f}_k - f_k)], \\ &\quad \max_{k\in K2}((f_k - f_k(x^*)) + (f_k(x^*) - f_k(x')))/(f_k - \bar{f}_k)] \\ &< \max[\max_{k\in K1}(\bar{f}_k - f_k(x^*))/(\bar{f}_k - f_k), \max_{k\in K2}(f_k - f_k(x^*))/(f_k - \bar{f}_k)] = S(x^*) \end{aligned}$$

is obtained.

It follows from (10), that $S(x') < S(x^*)$ and $f_k(x^*) \geq \bar{f}_k$, $k \in K2 \cup K3$, which contradicts to (8). Hence x^* is a weak efficient solution for problem (1)–(4).

Problem (5)–(7) can be stated as the following equivalent mixed integer convex nonlinear programming problem:

(11) $$\min \alpha$$

(12) $$(\bar{f}_k - f_k(x))/(\bar{f}_k - f_k) \leq \alpha,\ k \in K1,$$

(13) $$(f_k - f_k(x))/(f_k - \bar{f}_k) \leq \alpha,\ k \in K2,$$

(14) $$f_k(x) \geq \bar{f}_k,\ k \in K2 \cup K3,$$

(15) $$x \in X,$$

(16) $$\alpha - \text{arbitrary}.$$

When problem (5)–(7) has no solution, then problem (11)–(16) also has no solution. This is because both problems have the same original constraints. When problem (5)–(7) has a solution, then (11)–(16) has a solution and the optimal values of their objective functions are equal. The last statement is derived from the following lemma:

Lemma: The optimal values of the objective functions of problems (5)–(7) and (11)–(16) are equal.

The single-objective problem (11)–(16) has three desirable computational properties. The first is that the solution obtained in the previous iteration is a feasible solution for the single-objective problem of the current iteration. This facilitates algorithms for solving the single-objective integer programming problems because now there is a starting feasible solution. This is especially important for the approximate "tabu search" type algorithms.

The second property is that the feasible solutions of problem (11)–(16) obtained by using approximate algorithms for solving the single-objective problems lie near the nondominated frontier of the multiobjective problem (1)–(4). In many cases, the application of approximate integer algorithms (of tabu search type, for example) to solve of problem (11)–(16) will lead to near (weak) nondominated solutions quickly, thus reducing the waiting time for the dialogue with the DM. This is especially important during the initial iterations, when the DM is learning about the problem and the process. During the learning period, it is possible to interrupt the approximate single-objective algorithm and use the approximate solution obtained so far.

The third property is connected with the search strategy "great benefit - little loss". The solutions obtained along the reference direction, defined by the modified aspiration point and the last solution found, are comparatively close, which enables the decrease of DM's tension in the estimation and choice of a local prefered solution. In other words, the influence of the so called "limitted comparability" is reduced.

3. The proposed algorithm

The proposed algorithm is based on the reference direction approach that is designed to help a DM to find the most preferred solution comparatively quickly. This is achieved by reducing the number of iterations and the single objective problems that need to be solved to find the most preferred solution. Further, the use of an approximate procedure to solve the single objective integer problems speeds up its performance further. At each iteration the DM gives aspiration levels of the objective functions. The original multiobjective problem is reduced to a series of single objective convex integer problems. The DM evaluates the solutions and he/she decides whether to change the reference direction or to stop.

If the DM believes that the current nondominated solution is far from the most preferred solution, an approximate polynomial algorithm may be used to solve a new single objective convex nonlinear integer problem. This results in near (weak) nondominated solutions. The quality of the solutions depends on the algorithm used to solve the single objective problems. When the DM feels that current (weak) nondominated solution is close to the most preferred solution, he/she may use an exact algorithm to obtain the optimal solution of the current single objective problem. The current (weak) nondominated solution is used as a starting point in the exact algorithm. The search procedure continues until the most preferred solution is found. To solve a "large" multiple objective problem, the DM may use only an approximate algorithm to solve the single objective convex integer problems.

The steps of the proposed algorithm may be stated as follows:

Step 1. If an initial integer feasible solution is available for the problem (1)–(4), go to Step 2; otherwise, set $f_k = 0$, and $\bar{f}_k = 1$, $k \in K$. Solve problem (11)–)16) using an approximate algorithm to obtain a feasible initial solution. If the DM is satisfied with this solution, stop.

Step 2. Ask the DM to provide the new aspiration levels.

Step 3. Ask the DM to choose the type of the algorithm – exact or approximate. If the DM selects the exact algorithm, go to Step 5.

Step 4. Ask the DM to specify t – the maximal number of near (weak) nondominated solutions the DM wants to see along the reference direction at the current iteration. Go to Step 6.

Step 5. Solve problem (11–16). Show the (weak) nondominated or near (weak) nondominated solution obtained (if the computing process is interrupted) to the DM. If he/she approves this solution, stop; otherwise, go to Step 2.

Step 6. Solve problem (11–16). Present the t (if more than t solutions are obtained) near (weak) nondominated solutions to the DM. If the DM is satisfied with one of them, stop; otherwise, go to Step 2.

Remark 1. When the DM sets the aspiration levels in Step 2, it is important to partition the objective functions in three groups, depending on his/her choice of which objective functions to improve, which to worsen, and which cannot be weakened.

Remark 2. In Step 3, it is important that the DM knows that the choice of an exact algorithm leads to a nondominated integer solution but it takes longer time to obtain it. If an approximate algorithm is chosen, the time to solve the scalarizing problem is less but it may possibly deteriorate the quality of the solution.

Remark 3. If the DM chooses an approximate algorithm in Step 4, the DM may specify t – the number of near (weak) nondominated solutions he/she wants to evaluate at this iteration. This is possible because we compute only approximate solutions. Further, in the learning phase, it is useful for the DM to evaluate several solutions.

Remark 4. When using an exact algorithm in Step 5, problem (11)–(16) is solved in order to obtain a (weak) nondominated solution. If the DM decides that the solution time is too long, he may interrupt the computing process and evaluate the latest approximate solution.

The proposed algorithm for solving multiobjective convex nonlinear integer problems has some advantages. The number of the single objective problems solved is equal to the number of the aspiration points. The DM needs to define only the aspiration levels of the objective functions. The DM operates in the objective functions space because in most cases the objective functions have physical or economic aspect. The application of approximate algorithms to solve the single objective problems reduces the computational time, thus facilitating the dialogue of the DM. This may predispose the DM positively to the process of solving the multiple objective problems. The evaluation of several, though near (weak) nondominated solutions along the reference direction, enables the DM to learn faster. The use of an approximate "tabu search" type algorithm works well in narrow feasible regions (as for problem (11)–(16)) with known initial feasible solution that aids in finding rather good and in many cases optimal solutions to the scalarizing problems.

5. Illustrative example

For the sake of clarity and ease of understanding, we illustrate the proposed algorithm with a simple example where the objective function space is also the variable space. Consider the following:

$$\max \{ f_1(x) = x_1,\ f_2(x) = x_2 \}$$

subject to:

$$g_1(x) = x_1^2 + x_2^2 - 14x_1 - 14x_2 + 49 \le 0,$$

$$g_2(x) = -x_1 - x_2 + 16 \le 0,$$

$$x_1 \ge 0 - \text{integer},$$

$$x_2 \ge 0 - \text{integer}.$$

Let X denotes the feasible set and $(f_1, f_2) = (13, 10) = (x_1, x_2)$.

Suppose that the DM would like to increase the value of f_2 and is willing to lower the value of f_1. That is, the DM provides $(\bar{f}_1, \bar{f}_2) = (13, 10)$ as the aspiration point. The modified aspiration point is $(\tilde{f}_1, \tilde{f}_2) = (13, 17)$. The following problem, corresponding to (11)–(16), is solved to find a new solution:

$$\min \alpha$$

subject to

$$(13 - x_1)/7 \le \alpha,$$
$$(13 - x_2)/7 \le \alpha,$$
$$x_1 \ge 6,$$
$$(x_1, x_2) \in X.$$

The (weak) nondominated solutions are: $(f_1, f_2) = (11, 12) = (x_1, x_2)$ and $\alpha=5/7$; $(f_1, f_2) = (10, 13) = (x_1, x_2)$ and $\alpha=4/7$. Suppose the DM chooses solution $(f_1, f_2) = (10, 13)$ as a preferred one.

Suppose the DM now wants to increase the value of f_1, reduce the values of f_2 and provides $(\bar{f}_1, \bar{f}_2) = (14, 10)$ as the aspiration point. The modified aspiration point is $(\tilde{f}_1, \tilde{f}_2) = (14, 13)$.

To find a solution, we solve the following problem corresponding to (11)–(16)

$$\min \alpha$$

subject to

$$(14 - x_1)/4 \le \alpha,$$
$$(13 - x_2)/3 \le \alpha,$$
$$x_2 \ge 10,$$
$$(x_1, x_2) \in X.$$

Suppose now the DM wants to find an exact (weak) efficient solution. Using an exact algorithm, the (weak) nondominated solution is $(f_1, f_2) = (12, 11) = (x_1, x_2)$ and $\alpha = 2/3$. If this solution is the most preferred for the DM, the computational process terminates.

5. Concluding remarks

We have proposed an interactive algorithm based on the reference direction approach to solve multiple objective convex nonlinear integer programming problems.

The scalarizing problem (11)–(16) provides the opportunity to overcome some of the difficulties associated with the computational complexities of solving these problems with insignificant deterioration of the quality of solutions. Furthermore, it allows the realization of "no great benefit – little loss" strategy when the DM agrees to lose more, but which does not have to be, if more balanced solutions exist.

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Интерактивный алгоритм отправных направлений выпуклого нелинейного мультикритериального программирования

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(Р е з ю м е)

Обсуждается интерактивный алгоритм, предназначенный для решения нелинейных выпуклых задач мультикритериального программирования. В нем затруднения, связанные с вычислительной сложностью этих задач, преодолеваются на основе включения скаляризирующей задачи, связанной с рядом особеностей. Подход значительно помагает лице, принимающем решения и делает интерактивный диалог более легким. Предложенный алгоритм иллюстрован примером.