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An Interactive Algorithm of Objective Programming

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I. Introduction

At present there exists a large variety of learning-oriented interactive algorithms of the multiobjective mathematical programming [1,2,3,4]. This fact is related to the role of the decision making and the desire of the user or the decision-maker (DM) to control the whole process.

An interactive algorithm of the multiobjective convex programming and based on the reference direction methods [5, 6] which are an expansion of the reference point ones [7, 8, 9] is discussed in the paper.

The algorithm suggests a new scalarizing problem and a new user's behaviour aiming at the fulfillment of the following requirement [1] – achievement for a small number of algorithhm iterations of certainty in the DM that the solution chosen by him/her is the best (acceptable) or among the best.

This behaviour is determined by the proposed here possibility for the DM to assign an importance of given aspiration levels of some objective functions.

Thus a basic scalarizing problem and under choice a modified scalarizing one are solved at each iteration of the interactive algorithm. We can state that there is a greater probability for the solution of the modified problem to be accepted as a preferred one.

The more active participation of the DM in the decision process and the comparison of one or two current solutions with the current reference point predetermine a comparatively quick confidence and convergence to the most preferred solution.

II. Problem statement and preliminary considerations

The mathematical statement of a convex multiple objective problem is:

(A) $$\max f(x)$$

subject to the constraints:

(1) $$g_j(x)\leq 0, j\in J=(j=1,...,r),$$

(2) $$h_l(x)=0,\ l\in L=(l=1,...,s),$$

(3) $$x_k\geq 0,\ k\in K=(k=1,...,n).$$

Here $x=(x_1, x_2, ..., x_n)$ is a vector of decision variables, $x\in X\subset R^n$;

$f(x)=(f_1(x), f_2(x), ..., f_m(x))$ is a vector function, $f_i(x): R^n\to R$;

$f_i(x), i\in I$ are concave functions;

$g_j(x), j\in J$ are convex functions;

X– a set of feasible solutions which is determined by the group constraints (1), (2) and (3);

By "max" we mean that all objective functions (criteria) have to be maximized.

The formulation of each multiobjective programming problem suggests that there does not exist a solution which maximizes all objective functions simultaneously. It is required that efficient solutions x^* be found where the increase of some objective functions is improved only by sacrificing some other functions.

Definition 2.1. The vector $x^*\in X$ is called an efficient solution if there does not exist another solution $x\in X$, such that

(4) $$f_i(x)\geq f_i(x^*) \text{ for } i\in I$$

and with a strict inequality holding for at least one index i.

Definition 2.2. The vector $x^*\in X$ is called a weak efficient solution if there does not exist another solution $x\in X$, such that the (5) are true.

(5) $$f_i(x)>f_i(x^*) \text{ for each index } i\in I$$

Definition 2.3. The vector $f(x^*)$ corresponding to the (weak) efficient solution x^* is called a (weak) nondominated vector function /(weak) nondominated solution/.

We suggest a reference direction approach for solving of the convex multiple objective programming problem.

We use the following definitions which help for the description of the interactive algorithm:

Definition 2.4. Reference point – a point in the criteria space with co-ordinates which are the assigned by the DM aspiration levels of the separate criteria.

Definition 2.5. Reference direction – a direction defined as the difference between the reference point and the found solution at the previous iteration; describes a preferred change in the criteria space.

Definition 2.6. Preferred solution – (weak) nondominated solution, which the DM chooses as a more acceptable one at the present iteration.

Definition 2.7. Final (most preferred) solution – the preferred solution at the last iteration. It satisfies the DM in sense that he/she agrees with the received levels of the objective functions.

Substituting a single-objective problem:

The multiobjective problem (A) is solved at several iterations. At each one of them a scalarizing problem is solved, which is a problem of the single objective mathematical programming.

The interactive algorithm discussed in this paper is based on the use of the following scalarizing problem:

(B) $$\min\{s(x)=\max_{i\in I_1}(af_i^k-f_i(x))/(af_i^k-f_i^{k-1})+\max_{i\in I_2}(f_i(x)-af_i^k)/(af_i^k-f_i^{k-1})\}$$

subject to:

$$f_i(x)\geq f_i^{k-1} \text{ for } i\in I_1\cup I_3,$$
$$f_i(x)\leq f_i^{k-1} \text{ for } i\in I_2,$$
$$x\in X,$$

where af_i^k is the aspiration level of the objective function $f_i(x)$ of k-th iteration,

f_i^k – a value of the objective function $f_i(x)$ of k-th iteration,

I_1 – a set of indices of criteria which the DM wants to improve,

I_2 – a set of indices of criteria which the DM may agree to weaken,

I_3 – a set of indices of criteria which the DM accepts in the way they are.

The aspiration level af_i^k is assigned in such a way that:

(6) $$af_i^k > f_i^{k-1} \text{ for } i\in I_1$$
$$af_i^k < f_i^{k-1} \text{ for } i\in I_2$$
$$af_i^k = f_i^{k-1} \text{ for } i\in I_3,\ I=I_1\cup I_2\cup I_3.$$

The scalarizing function minimizes the difference between the biggest standardized difference between the aspiration levels and the values of the functions with indices $i○I1$ and the biggest standardized difference between the values of the functions with indices $i○I2$ and their aspiration levels.

Statement:

The optimum solution of (B) is a weak efficient solution of (A).

Proof: Let x^* is an optimum solution of the (B) problem. Then

(7) $$s(x)\geq s(x^*) \text{ for each } x\in X.$$

Let x^* is not a weak efficient solution for (A). Then (according to def. 2.2) there exists a point $x'\in X$ for which $f_i(x')>f_i(x^*)$ for each $i\in I$. Now

$$s(x')=\max_{i\in I_1}(af_i^k-f_i(x'))/(af_i^k-f_i^{k-1})+\max_{i\in I_2}(f_i(x')-af_i^k)/(af_i^k-f_i^{k-1})=$$
$$=\max_{i\in I_1}((af_i^k-f_i(x^*))+(f_i(x^*)-f_i(x')))/(af_i^k-f_i^{k-1})+$$
$$+\max_{i\in I_2}((f_i(x^*)-af_i^k)+(f_i(x')-f_i(x^*)))/(af_i^k-f_i^{k-1})<$$
$$<\max_{i\in I_1}(af_i^k-f_i(x^*))/(af_i^k-f_i^{k-1})+$$
$$+\max_{i\in I_2}(f_i(x^*)-af_i^k)/(af_i^k-f_i^{k-1})=s(x^*)$$

i.e.

(8) $$s(x')<s(x^*).$$

As (8) contradicts to (7), x^* is a weak efficient solution of (A).

The minimization of the objective function of (B) can be reduced to the following equivalent problem:

(C) $$\min(\alpha+\beta)$$

subject to constraints:

$$(af_i^k - f_i(x))/(af_i^k - f_i^{k-1}) \le \alpha \text{ for } i \in I_1,$$
$$(f_i(x) - af_i^k)/(af_i^k - f_i^{k-1}) \le \beta \text{ for } i \in I_2,$$
$$f_i(x) \ge f_i^{k-1} \text{ for } i \in I_1 \cup I_3,$$
$$f_i(x) \le f_i^{k-1} \text{ for } i \in I_2,$$
$$x \in X;$$

where α and β are scalars.

Lemma. The optimum values of the objective functions of (B) and (C) are equal, i.e.

$$\min_{x \in X} (\alpha + \beta) = \min_{x \in X} \{\max_{i \in I_1} (af_i^k - f_i(x))/(af_i^k - f_i^{k-1}) +$$
$$+ \max_{i \in I_2} (f_i(x) - af_i^k)/(af_i^k - f_i^{k-1})\}.$$

Proof. As $\alpha \ge (af_i^k - f_i(x))/(af_i^k - f_i^{k-1})$ holds for each $i \in I1$, it follows that

(9) $$\alpha \ge \max_{i \in I_1} (af_i^k - f_i(x))/(af_i^k - f_i^{k-1}).$$

By analogy with $\beta \ge (af_i^k - f_i(x))/(af_i^k - f_i^{k-1})$ for each $i \in I_2$, it follows that

(10) $$\beta \ge \max_{i \in I_2} (f_i(x) - af_i^k)/(af_i^k - f_i^{k-1}).$$

We sum up the left and right members of the inequalities. Then

$$\alpha + \beta \ge \max_{i \in I_1} (af_i^k - f_i(x))/(af_i^k - f_i^{k-1}) + \max_{i \in I_2} (f_i(x) - af_i^k)/(af_i^k - f_i^{k-1}).$$

Let x^* is the optimum solution of the equivalent problem. Then

$$\min_{x \in X}(\alpha + \beta) = \max_{i \in I_1} (af_i^k - f_i(x^*))/(af_i^k - f_i^{k-1}) + \max_{i \in I_2} (f_i(x^*) - af_i^k)/(af_i^k - f_i^{k-1}).$$

But as

$$\{\max_{i \in I_1} (af_i^k - f_i(x^*))/(af_i^k - f_i^{k-1}) + \max_{i \in I_2} (f_i(x^*) - af_i^k)/(af_i^k - f_i^{k-1})\} =$$
$$= \min_{x \in X} \{\max_{i \in I_1} (af_i^k - f_i(x))/(af_i^k - f_i^{k-1}) + \max_{i \in I_2} (f_i(x) - af_i^k)/(af_i^k - f_i^{k-1}))\},$$

it follows that

$$\min_{x \in X}(\alpha + \beta) = \min_{x \in X} \{\max_{i \in I_1} (af_i^k - f_i(x))/(af_i^k - f_i^{k-1}) + \max_{i \in I_2} (f_i(x) - af_i^k)/(af_i^k - f_i^{k-1})\}.$$

Since the problems (B) and (C) are equivalent the optimum solution of (C) is a weak efficient solution of (A).

III. Proposed algorithm

The proposed algorithm follows the common scheme of the interactive methods of the reference directions. Information from the DM regarding the aspiration levels of the different criteria is required for the formulation of the scalarizing problem (B). If the DM doesn't choose criteria whose values he/she wishes to improve the algorithm stops. The starting point for the single objective problem is one feasible solution of the multiobjective problem. The initial weak efficient solution (the solution at the first iteration) is defined under the assumption that the DM wishes an improvement of all criteria. The scalarizing problem (B) without the constraints $f_i(x) \geq f_i^0$ for $i \in I$ is solved. At each next iteration a starting point for the scalarizing problem is the preferred solution from the previous iteration. The algorithm stops at the finding of the most preferred solution.

The described algorithm includes a new element in its construction.

The aim is that the DM be given a greater freedom in the control of the process for a search for a more acceptable solution and a possibility to orient himself for the solutions of the multiobjective problem under the chosen reference point. It is suggested that the DM analyze the found solution after having solved the scalarizing problem and if this solution doesn't satisfy him/her to look for another one for the chosen reference point by solving a modified scalarizing problem. We designate these problems as a basic scalarizing problem and an auxiliary scalarizing one.

The basic scalarizing problem projects the reference point over the nondominated set of points. At each iteration the DM expresses also his/her preferences for importance of the criteria together with choosing the criteria, whose values will be improved and those ones whose values will be worsened. But taking into consideration the common balance among the levels of the objective functions in the found solution, the DM is likely to be not satisfied with some of them. Then the DM is suggested that he/she rank the given aspiration levels by importance and look for another nondominated solution. It should satisfy the requirements of the DM about the specified criteria by him which are with bigger importance of the aspiration levels by receiving values of these criteria bigger or equal to their aspiration levels. When these requirements have been met the probability that the DM agree with the other received criteria values also and therefore accept the solution as final is big. An auxiliary scalarizing problem is solved. There exist two possibilities after the solution of the base problem has been found:

– The DM doesn't wish worsening below the aspiration levels of the objective functions (marked with indices $i \in I_i \subseteq I_2$) with received undesired values.

An auxiliary problem (H1) is solved which is the basic one with a group of additional constraints:

$$f_i(x) \geq af_i^k,\ i \in I_i \subseteq I_2.$$

– The DM wishes an achievement of the aspiration levels of the objective functions (marked with indices $i \in I_j \text{H} I1$) with received undesired values.

An auxiliary problem (H2) is solved which is the basic one with a group of additional constraints:

$$f_i(x) \geq af_i^k,\ i \in I_j \subseteq I_1.$$

Steps of the algorithm:

Step 1. Finding an initial weak efficient solution and the respective weak nondominated vector.

Step 2. Submission of the solution to the DM for estimation. If the DM decides that the solution satisfies him then Step 10 comes. Otherwise the DM accepts the solution as a preferred one at the present iteration and goes to Step 3.

Step 3. The DM defines a new reference point. The equivalent problem (C) of the basic scalarizing problem (B) is solved by taking into consideration the preferred solution from the previous iteration.

Step 4. Estimation of the found solution by the DM. If it satisfies the DM then step 10 comes. Otherwise he/she goes to Step 5.

Step 5. If the DM accepts the solution for a preferred one at the present iteration then Step 3 comes. Otherwise Step 6 comes.

Step 6. If the DM points objective functions with indices $I_i \subseteq I_1$ the auxiliary scalarizing problem H1 is solved. A transition to Step 8 comes.

Step 7. If the DM points objective functions with indices $I_j \subseteq I_2$ the auxiliary scalarizing problem H1 is solved. The problem has a solution under feasible aspiration levels and then a transition to Step 8 follows. If the aspiration levels are not feasible then the DM accepts the solution of the basic scalarizing problem (B) for a preferred one at the present iteration and goes to Step 3.

Step 8. Estimation of the found solution by the DM. If it satisfies the DM a transition to Step 10 comes. Otherwise Step 9 follows.

Step 9. Choice of the more acceptable solution between the solution of problem (B) and the solution of problem (H1) or (H2) for a preferred one at the present iteration. Transition to Step 3.

Step 10. End of the algorithm. The found solution is the most preferred one.

It is very important that the aspiration levels of the improved criteria can be infeasible in some cases. The algorithm can provide additional information for helping the DM in his/her decision to refer to the auxiliary problem. That information refers to a check-up of the feasibility of the pointed aspiration levels. On the other side it is possible that these levels be achieved by worsening the objective functions with indices $i \in I_2$ under their aspiration levels. But in this case a minimum risk is taken which is expressed in the minimum taking away of the values of the worsened objective functions from the desired values.

IV. Example

The problem is stated as:

$$\max f_1 = -(x_1 - 4)^2 - (x_2 - 3)^2,$$
$$\max f_2 = -x_1^2 - 9(x_2 - 3)^2,$$
$$\max f_3 = -(x_1 + 0,5)^2 - (x^2 + 1)^2,$$

subject to:

$$4x_1^2 + 9x_2^2 - 36 \le 0,$$
$$(x_1 - 1)^2 + (x_2 + 3)^2 - 20,25 = 0.$$

Starting feasible vector for the problem (B):

$$(x_1^0, x_2^0) = (0, 0,5); (f_1^0, f_2^0, f_3^0) = (-18,25, -20,25, -6,5).$$

Iteration 1:

$$I_1 = \{1, 2, 3\} \text{ and } (af_1^1, af_2^1, af_3^1) = (-12, -17, -4).$$

The solution of the basic scalarizing problem is:

$$(x_1^2, x_2^2)^b = (0,54088, 1,47652); (f_1^1, f_2^1, f_3^1)^b = (-14,2865, -21,1815, -7,21657).$$

Iteration 2:

$$I_1 = \{1\}, I_2 = \{2, 3\} \text{ and } (af_1^2, af_2^2, af_3^2) = (-13,5, -22, -7,9),$$
$$(x_1^2, x_2^2)^b = (1,68248, 1,44795); (f_1^2, f_2^2, f_3^2)^b = (-7,77976, -24,5106, -10,7557).$$

As the DM does not wish an worsening of the objective function $f_2(x)$ under the given aspiration level he/she solves an auxiliary problem (H1) with the constraint $f_2(x) \ge af_2^2$.

The solution of the auxiliary scalarizing problem is:

$$(x_1^2, x_2^2)^{h1} = (1,24986, 1,49306); (f_1^2, f_2^2, f_3^2)^{h1} = (-9,83417, -22, -9,27733).$$

Let this solution does not satisfy the DM also but it is accepted for a preferred one at the present iteration.

Iteration 3:

$I_1=\{3\}$, $I_2=\{1\}$, $I_3=\{2\}$ and $(af_1^3, af_2^3, af_3^3) = (-12,8, -22, -8)$.

The solution of problem (B) is

$(x_1^3, x_2^3)^b = (0,796071, 1,49538)$; $(f_1^3, f_2^3, f_3^3)^b = (-12,5291, -21,0087, -7,90671)$.

It satisfies the DM, who accepts it as a final one of the multiobjective problem.

V. Conclusion

The main idea in the development of the algorithm is that the DM should find the final solution with a small number of iterations of the multiobjective problem.

The following main characteristics of the reference direction methods help for this:

– the finding of the ideal point is not required which leads to a reduction of the number of the solved single-objective problems,

– the intermediate solutions of the multiobjective problem are weak efficient points which allows each of them to be suggested to the DM for an estimation as the most preferred solution.

In parallel with this the efficiency of the process for searching for the final solution and expressed in the achievement of certainty for the most acceptable solution for a small number of iterations depends on the behaviour of the DM. The described algorithm discusses a new user behaviour. By solving a basic problem and an auxiliary scalarizing one at each iteration the DM has the possibility to express his/her preferences not only for importance of the criteria but also for importance of some aspiration levels. Then the probability for the solution of the auxiliary to be accepted as a final one is bigger than the same probability for the basic problem. It is important to note that the requirement for the achievement of given aspiration levels exists when a series of real problems are solved.

References

1. Gardiner, L., D. Vanderpooten. Interactive Multiple Criteria Procedures: Some Reflections. Multicriteria Analysis (J. Climaco – ed.). Berlin, Springer Verlag, 1997, 290–301.
2. Shing, W., A. Ravindran. Interactive multiple objective optimization: survey 1 – continuous case. – In: Intern. J. of Computers and Operations Research, **18**, 1991, No1, 97–114.
3. Vincke, Ph. Multicriteria Decision Aid. Wiley & Sons ltd, 1992.
4. Steuer, R. E. Multiple Criteria Optimization: Theory, Computation, and Application. New York, Wiley, 1986.
5. Narula, S., L. Kirilov, V. Vassilev. Reference direction approach for solving multiple objective nonlinear problems. – IEEE Transactions on System, Man, and Cybernetics, **24**, 1994, No5, 804–806.
6. Narula, S., L. Kirilov, V. Vassilev. An Interactive Algorithm for Solving Multiple Objective Nonlinear Programming Problems-Multiple Criteria Decision Making (G. Tzeng, ed). Springer Verlag, Berlin, 1994, 119–127.
7. Korhonen, P., J. Laakso. A visual interactive method for solving the multiple criteria problems. – European Journal of Operational Research, **24**, 1986, 277– 287.
8. Wierzbicki, A.P. A mathematical basis for satisfying decision-maker. – Mathematical Modeling, 1982, No3, 391–405.
9. Nakayama, H. Sensitivity and trade-off analysis in multiobjective programming. – LNEMS, **337**, 1989, 86–93.

Алгоритм многокритериального выпуклого программирования

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(Р е з ю м е)

Предлагается интерактивный алгоритм выпуклого многокритериального математического программирования, использующего подход отправных направлений. Алгоритм базируется на специфической скаляризирующей задаче и ориентированное потребительское поведение в процессе принятия решения. В каждой итерации решается скаляризирующая, а при выборе – и модифицированная скаляризирующая задача, что приводит к более быстрой сходимости к финальному решению.