A family of symmetric polynomials of the eigenvalues of a matrix

Vladimir V. Monov*
Institute of Information Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2
1113 Sofia, Bulgaria

Abstract. In this paper, we consider a family of symmetric polynomials of the eigenvalues of a complex matrix A and find an explicit expression of each member of the family as a polynomial of the entries of A with positive coefficients. In the case of a nonnegative matrix, one immediately obtains a family of inequalities involving matrix eigenvalues and diagonal entries. Equivalent forms of some of the obtained results as well as connections with known results and specific applications are also presented. In the concluding part of the paper, we provide comments and conjecture further inequalities related with nonnegative matrices.

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1 Introduction

Let $A = [a_{ij}] \in M_n(\mathbb{C})$ and $N = \{1, 2, \ldots, n\}$. For $\alpha \subseteq N$, $\langle\alpha\rangle$ will denote the cardinality of α and by $A[\alpha]$ we shall denote the principal submatrix of A in rows and columns indexed by α. The multiset of eigenvalues $\lambda_1, \ldots, \lambda_n$ of A (spectrum of A) will be denoted by $\Lambda = (\lambda_1, \ldots, \lambda_n)$. In the next section,we shall consider polynomials of $\lambda_1, \ldots, \lambda_n$ and when no matrix is specified, $\lambda_1, \ldots, \lambda_n$ will be interpreted as independent variables. The notation $A \geq 0$ ($A > 0$) will be used if $A = [a_{ij}] \in M_n(\mathbb{R})$ with $a_{ij} \geq 0$ ($a_{ij} > 0$,) $i, j = 1, \ldots, n$.

Several types of symmetric polynomials of the eigenvalues of a square matrix are well known and play an important role in matrix theory. Typical examples are the elementary symmetric polynomials

$$e_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k},\ k = 0, 1, \ldots, \tag{1.1}$$

where $e_0 = 1$ and $e_k = 0$ for $k > n$, and the power sums

$$s_k = \sum_{i=1}^{n} \lambda_i^k,\ k = 0, 1, \ldots. \tag{1.2}$$

Polynomials e_k and s_k are related with matrix entries by the equalities

$$e_k = \sum_{\substack{\alpha \subseteq N \\ \langle\alpha\rangle = k}} \det A[\alpha] \text{ and } s_k = \mathrm{tr}\,(A^k),\ k = 0, 1, \ldots \tag{1.3}$$

*E-mail: vmonov@iit.bas.bg; Tel: (+359 2) 9792485; Fax: (+359 2) 8720497

which are widely used in various matrix theoretic contexts.

Apart from their importance in the study of certain matrix properties, symmetric polynomials appear in the representation theory of symmetric groups, algebraic and analytic combinatorics, mathematical physics, etc. It is well known [6] that both sets of polynomials (1.1) and (1.2) represent natural choices for constructing bases of the ring $\Lambda_{n,\mathbb{Z}}$ of all symmetric polynomials of $\lambda_1, \ldots, \lambda_n$ with coefficients in $\mathbb{Z}$. Thus, any element of $\Lambda_{n,\mathbb{Z}}$ can be uniquely written as a polynomial in the elementary symmetric polynomials or a polynomial in the power sums which, in view of (1.3), implies that any symmetric polynomial in the eigenvalues of a matrix can be written in terms of matrix entries. However, finding the explicit form in which an element of $\Lambda_{n,\mathbb{Z}}$ is written as a linear combination of the elements of a given basis is not always straightforward and the study of relationships among the various bases of $\Lambda_{n,\mathbb{Z}}$ is an important subject in the theory of symmetric functions.

In this paper, we define a family of symmetric polynomials of the eigenvalues of a matrix $A \in M_n(\mathbb{C})$ and obtain an expression of each member of the family as a sum of products of the entries of A. The polynomial family includes as special cases the power sums and the complete homogeneous symmetric polynomials of matrix eigenvalues. Our approach is based on analytical tools involving scalar and matrix power series expansions. A connection with the classical MacMahon's Master Theorem from enumerative combinatorics is also pointed out. In the special case when A is a nonnegative matrix, one immediately obtains a family of inequalities relating the eigenvalues and diagonal entries of A. Some other applications of the obtained results are given in addition. In the concluding part of the paper, we provide comments and conjecture further inequalities related with nonnegative matrices.

2 Main Result

Given integers m and n, $1 \leq m \leq n$, let $Q_{m,n}$ be the set of all strictly increasing sequences of the form $\mathbf{i} = (i_1, \ldots, i_m)$ where $i_1, \ldots, i_m$ are elements from the set N. Clearly, $Q_{m,n}$ consists of $\binom{n}{m}$ sequences. If $\mathbf{j} = (j_1, \ldots, j_m)$ is a sequence of nonnegative integers, $|\,\mathbf{j}\,|$ will denote the sum $|\,\mathbf{j}\,| = j_1 + \ldots + j_m$. The order of elements in $\mathbf{j} = (j_1, \ldots, j_m)$ will matter so that the number of all ordered nonnegative sequences satisfying $|\,\mathbf{j}\,| = k$ for some nonnegative integer k, is $\binom{k+m-1}{k}$. The complete homogeneous symmetric polynomial of degree k in m independent variables $x_1, \ldots, x_m$ will be denoted by

$$h_k(x_1, \ldots, x_m) = \sum_{|\,\mathbf{j}\,|=k} x_1^{j_1} x_2^{j_2} \ldots x_m^{j_m}. \tag{2.1}$$

For a matrix $A \in M_n(\mathbb{C})$, the principal submatrix obtained by deleting rows and columns of A with indexes $i_1, \ldots, i_m$ is denoted by $A(i_1, \ldots, i_m)$. A special notation is used for the diagonal entries of a matrix, i.e., $[A]_p$ denotes the diagonal element of A at position (p, p). This notation is particularly suitable in identifying diagonal elements of powers of a given submatrix of A. Thus, $[A(i_1, \ldots, i_m)^q]_p$ denotes the diagonal element at position (p, p) of the $q-$th power of $A(i_1, \ldots, i_m)$; note that, in this case, p ranges from 1 to $n - m$.

Now, let $A \in M_n(\mathbb{C})$ be given with spectrum $\Lambda = (\lambda_1, \ldots, \lambda_n)$. For each m, $1 \leq m \leq n$, we shall consider polynomials of $\lambda_1, \ldots, \lambda_n$ given by

$$s_{k,m}(\Lambda) = \sum_{\mathbf{i} \in Q_{m,n}} h_k(\lambda_{i_1}, \ldots, \lambda_{i_m}), \;\; k = 0, 1, \ldots. \tag{2.2}$$

Note that the special cases $m = 1$ and $m = n$ reduce to $s_{k,1}(\Lambda) = s_k$ and $s_{k,n}(\Lambda) = h_k(\lambda_1, \dots, \lambda_n)$, $k = 0, 1, \dots$. We shall also define a family of polynomials of the entries of A. In particular, given m, $1 \le m \le n$, let $p_{k,m}(A)$ denote

$$p_{k,m}(A) = \sum_{\mathbf{i} \in Q_{m,n}} r_k(A; i_1, \dots, i_m) \tag{2.3}$$

where

$$r_k(A; i_1, \dots, i_m) = \sum_{|\,\mathbf{j}\,|=k} [A^{j_1}]_{i_1} [A(i_1)^{j_2}]_{i_2-1} \dots [A(i_1, \dots, i_{m-1})^{j_m}]_{i_m-m+1} \tag{2.4}$$

for $k = 0, 1, \dots$. Thus, $s_{k,m}(\Lambda)$ is a homogeneous symmetric polynomial of degree k in the eigenvalues of A while $p_{k,m}(A)$ is a homogeneous polynomial of the same degree in the entries of A. Both $s_{k,m}(\Lambda)$ and $p_{k,m}(A)$ are polynomials with positive integer coefficients. From (2.1)-(2.4), it can be easily seen that if A is a triangular matrix then $s_{k,m}(\Lambda) = p_{k,m}(A)$. We shall prove this equality in the general case.

Theorem 2.1 *Let $A \in M_n(\mathbb{C})$ and $\Lambda = (\lambda_1, \dots, \lambda_n)$ be the spectrum of A. For each m, $1 \le m \le n$,*

$$s_{k,m}(\Lambda) = p_{k,m}(A), \;\; k = 0, 1, \dots. \tag{2.5}$$

Proof. Given m, $1 \le m \le n$, we shall construct two power series with coefficients $s_{k,m}(\Lambda)$ and $p_{k,m}(A)$, $k = 0, 1, \dots$, respectively, and show that both series represent expansions of one and the same function. Let

$$f(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \det(\lambda I - A) \tag{2.6}$$

and define $F_m(\lambda)$ as

$$F_m(\lambda) = \frac{\lambda^m}{m!} \frac{f^{(m)}(\lambda)}{f(\lambda)} \tag{2.7}$$

where $f^{(m)}(\lambda)$ is the m-th derivative of $f(\lambda)$, i.e. $f^{(m)}(\lambda) = d^m f(\lambda)/d\lambda^m$, $1 \le m \le n$. By differentiating m times the product in (2.6), it is obtained

$$F_m(\lambda) = \lambda^m \sum_{\mathbf{i} \in Q_{m,n}} (\lambda - \lambda_{i_1})^{-1} \dots (\lambda - \lambda_{i_m})^{-1}. \tag{2.8}$$

For values of λ satisfying $|\,\lambda\,| > \max_{1 \le i \le n} |\,\lambda_i\,|$, each term of the product in the right hand side of (2.8) expands in a geometric series so that after multiplication, we have

$$\begin{aligned} F_m(\lambda) \;\; &= \;\; \sum_{\mathbf{i} \in Q_{m,n}} \sum_{k=0}^{\infty} h_k(\lambda_{i_1}, \dots, \lambda_{i_m}) \lambda^{-k} \\ &= \;\; \sum_{k=0}^{\infty} s_{k,m}(\Lambda) \lambda^{-k}. \end{aligned} \tag{2.9}$$

On the other hand, differentiating the determinant function in (2.6) by applying m-times the differentiation formula [5]

$$\frac{d}{d\lambda} \det(\lambda I - A) = \sum_{i=1}^{n} \det(\lambda I - A)(i), \tag{2.10}$$

it is obtained

$$\begin{aligned} F_m(\lambda) &= \frac{\lambda^m}{\det(\lambda I - A)} \sum_{\mathbf{i}\in Q_{m,n}} \det(\lambda I - A)(i_1, \ldots, i_m) \\ &= \lambda^m \sum_{\mathbf{i}\in Q_{m,n}} \frac{\det(\lambda I - A)(i_1)}{\det(\lambda I - A)} \frac{\det(\lambda I - A)(i_1, i_2)}{\det(\lambda I - A)(i_1)} \cdots \frac{\det(\lambda I - A)(i_1, \ldots, i_m)}{\det(\lambda I - A)(i_1, \ldots, i_{m-1})}. \end{aligned} \tag{2.11}$$

It is easily seen that for $|\lambda| > \max_{1\le i\le n} |\lambda_i|$, each factor of the product in (2.11) is a diagonal element in the inverse of some principal submatrix of $(\lambda I - A)$. By expanding these inverse matrices in matrix power series and taking the respective diagonal entries, we get

$$\begin{aligned} \frac{\det(\lambda I - A)(i_1)}{\det(\lambda I - A)} &= \lambda^{-1} \sum_{k=0}^{\infty} [A^k]_{i_1} \lambda^{-k} \\ \frac{\det(\lambda I - A)(i_1, i_2)}{\det(\lambda I - A)(i_1)} &= \lambda^{-1} \sum_{k=0}^{\infty} [A(i_1)^k]_{i_2-1} \lambda^{-k} \\ &\vdots \\ \frac{\det(\lambda I - A)(i_1, \ldots, i_m)}{\det(\lambda I - A)(i_1, \ldots, i_{m-1})} &= \lambda^{-1} \sum_{k=0}^{\infty} [A(i_1, \ldots, i_{m-1})^k]_{i_m-m+1} \lambda^{-k}. \end{aligned} \tag{2.12}$$

Substituting (2.12) into (2.11) and multiplying gives

$$\begin{aligned} F_m(\lambda) &= \sum_{\mathbf{i}\in Q_{m,n}} \sum_{k=0}^{\infty} r_k(A; i_1, \ldots, i_m) \lambda^{-k} \\ &= \sum_{k=0}^{\infty} p_{k,m}(A) \lambda^{-k}. \end{aligned} \tag{2.13}$$

Since the left hand sides of (2.9) and (2.13) agree, a comparison of coefficients in the right hand sides implies equality (2.5).

Given a matrix $A \in M_n(\mathbb{C})$, it follows from (2.5) that $p_{k,m}(A)$ is invariant under a similarity transformation of A, i.e. $p_{k,m}(A) = p_{k,m}(S^{-1}AS)$ for any nonsingular matrix S. Since AB and BA have the same spectrum for any $A, B \in M_n(\mathbb{C})$, it also follows from (2.5) that

$$p_{k,m}(AB) = p_{k,m}(BA). \tag{2.14}$$

Conversely, from (2.14) one can easily obtain (2.5) by using the Schur triangularization theorem and the fact that equalities (2.5) are obviously satisfied for a triangular matrix.

The following result is a consequence of Theorem 2.1 concerning nonnegative matrices.

Corollary 2.1 *Let $A = [a_{ij}] \in M_n(\mathbb{R})$, $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and $A \ge 0$. For each m, $1 \le m \le n$,*

$$s_{k,m}(\Lambda) \ge \sum_{\mathbf{i}\in Q_{m,n}} h_k(a_{i_1 i_1}, \ldots, a_{i_m i_m}), \; k = 0, 1, \ldots. \tag{2.15}$$

Proof. Let $diag(A)$ denote the diagonal matrix with diagonal entries $a_{11}, \ldots, a_{nn}$. For any nonnegative matrix $B = [b_{ij}] \in M_n(\mathbb{R})$, we have $[B^k]_i \ge b_{ii}^k$ for $i = 1, \ldots, n$ and

$k = 0, 1, \ldots$. By applying this inequality in (2.4), it is easily seen that

$$p_{k,m}(A) \geq p_{k,m}(diag(A)) = \sum_{\mathbf{i} \in Q_{m,n}} h_k(a_{i_1 i_1}, \ldots, a_{i_m i_m}) \tag{2.16}$$

and thus, (2.15) follows from (2.5) and (2.16).

Obviously, in the simplest special case $m = 1$ Theorem 2.1 yields $s_k = \operatorname{tr}(A^k)$ and Corollary 2.1 reduces to the inequalities $s_k \geq \sum_{i=1}^n a_{ii}^k$, $k = 0, 1, \ldots$.

In what follows, we consider polynomials (2.2) in the cases $m = 2$ and $m = n$. For $m = n$, we have $s_{k,n}(\Lambda) = h_k(\lambda_1, \ldots, \lambda_n)$ and in this case, we shall use the celebrated MacMahon's Master Theorem [7] in order to find another expression of $s_{k,n}(\Lambda)$ in terms of the entries of A.

Theorem 2.2 (MacMahon's Master Theorem) *Let $A = [a_{ij}] \in M_n(\mathbb{C})$, $X = diag[x_1, \ldots, x_n]$ and $X_i = a_{i1}x_1 + \ldots + a_{in}x_n$, $i = 1, \ldots, n$. The coefficient of the term $x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$ in the power series expansion of $1/\det(I - AX)$ is equal to the coefficient of the term $x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$ in the expansion of the product $X_1^{j_1} X_2^{j_2} \ldots X_n^{j_n}$.*

The above theorem is an important result in enumerative combinatorics having a number of applications and various extensions and generalizations. In relation with this result, we note that

$$\begin{aligned} \det(I - AX) &= \sum_{\alpha \subseteq N} (-1)^{\langle \alpha \rangle} \det A[\alpha] \det X[\alpha] \\ &= 1 - \sum_{\emptyset \neq \alpha \subseteq N} (-1)^{\langle \alpha \rangle - 1} \det A[\alpha] \det X[\alpha] \end{aligned} \tag{2.17}$$

and thus, the power series expansion of $1/\det(I - AX)$ is given by

$$\frac{1}{\det(I - AX)} = \frac{1}{1 - \Sigma} = 1 + \Sigma + \Sigma^2 + \ldots \tag{2.18}$$

where Σ is the sum in the right hand side of (2.17).

We shall apply Theorem 2.2 by using the following notation. Given a series $P(x_1, \ldots, x_n)$ in powers of $x_1, \ldots, x_n$, $c(P(x_1, \ldots, x_n) : x_1^{j_1} \ldots x_n^{j_n})$ will denote the coefficient of $x_1^{j_1} \ldots x_n^{j_n}$ in $P(x_1, \ldots, x_n)$. In this notation, MacMahon's Master Theorem states that

$$c\left(\frac{1}{\det(I - AX)} : x_1^{j_1} \ldots x_n^{j_n}\right) = c(X_1^{j_1} \ldots X_n^{j_n} : x_1^{j_1} \ldots x_n^{j_n}). \tag{2.19}$$

Now, by substituting $x_1 = \ldots = x_n = x$ in the power series expansion (2.18), it is easily seen that

$$c\left(\frac{1}{\det(I - xA)} : x^k\right) = \sum_{|\mathbf{j}| = k} c\left(\frac{1}{\det(I - AX)} : x_1^{j_1} \ldots x_n^{j_n}\right). \tag{2.20}$$

On the other hand

$$\frac{1}{\det(I - xA)} = \prod_{i=1}^n (1 - x\lambda_i)^{-1} = \sum_{k=0}^\infty h_k(\lambda_1, \ldots, \lambda_n) x^k \tag{2.21}$$

and, hence, by (2.19), (2.20) and (2.21) it is obtained

$$h_k(\lambda_1, \ldots, \lambda_n) = \sum_{|\mathbf{j}| = k} c(X_1^{j_1} \ldots X_n^{j_n} : x_1^{j_1} \ldots x_n^{j_n}). \tag{2.22}$$

With $X_1, \ldots, X_n$ as defined in Theorem 2.2, equality (2.22) gives an alternative expression of $s_{k,n}(\Lambda) = h_k(\lambda_1, \ldots, \lambda_n)$ obtained by means of this theorem.

Next, we state an explicit formula for $s_{k,2}(\Lambda)$ as a polynomial of the power sums $s_0, \ldots, s_k$ for $k = 0, 1, \ldots$ and show some applications of this result.

Proposition 2.1 *Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$, $s_{k,2}(\Lambda)$ be given by (2.2) with $m = 2$ and s_k denote the power sums (1.2). Then*

$$s_{k,2}(\Lambda) = \frac{1}{2}\left(\sum_{p+q=k} s_p s_q - (k+1)s_k\right), \quad k = 0, 1, \ldots. \tag{2.23}$$

Proof. The sequence s_k, $k = 0, 1, \ldots$ is defined by the generating function

$$\sum_{k=0}^{\infty} s_k x^k = \sum_{i=1}^{n} \frac{1}{1 - x\lambda_i} \tag{2.24}$$

and also, for any λ_{i_1} and λ_{i_2}, the sequence $h_k(\lambda_{i_1}, \lambda_{i_2})$, $k = 0, 1, \ldots$ can be given by

$$\sum_{k=0}^{\infty} h_k(\lambda_{i_1}, \lambda_{i_2}) x^k = \frac{1}{1 - x\lambda_{i_1}} \frac{1}{1 - x\lambda_{i_2}}. \tag{2.25}$$

By summing the left and right sides of (2.25) over all $\mathbf{i} = (i_1, i_2) \in Q_{2,n}$ and taking into account (2.2), it is obtained

$$\begin{aligned}
\sum_{k=0}^{\infty} s_{k,2}(\Lambda) x^k &= \sum_{\mathbf{i} \in Q_{2,n}} \frac{1}{1 - x\lambda_{i_1}} \frac{1}{1 - x\lambda_{i_2}} \\
&= \frac{1}{2}\left(\left(\sum_{i=1}^{n} \frac{1}{1 - x\lambda_i}\right)^2 - \sum_{i=1}^{n}\left(\frac{1}{1 - x\lambda_i}\right)^2\right).
\end{aligned} \tag{2.26}$$

Using the power series expansion

$$\left(\frac{1}{1 - x\lambda_i}\right)^2 = \sum_{k=0}^{\infty} (k+1)\lambda_i^k x^k, \; i = 1, \ldots, n \tag{2.27}$$

it follows from (2.24) and (2.26) that

$$\sum_{k=0}^{\infty} s_{k,2}(\Lambda) x^k = \frac{1}{2}\left(\left(\sum_{k=0}^{\infty} s_k x^k\right)^2 - \sum_{k=0}^{\infty} (k+1) s_k x^k\right). \tag{2.28}$$

By equating the coefficients of the equal powers of x in both sides of (2.28), we obtain (2.23).

In the rest of this section, we use equality (2.23) in order to establish a trace property of the bialternate product of matrices and to obtain a relation between the power sums of the zeros of a polynomial and the power sums of the zeros of its first derivative.

The bialternate product of $A = [a_{ij}]$ and $B = [b_{ij}]$, $i, j = 1, \ldots, n$ is defined as $F = A \cdot B$ where the entries of F are given by

$$f_{pq,rs} = \frac{1}{2}\left(\det\begin{bmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{bmatrix} + \det\begin{bmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{bmatrix}\right) \tag{2.29}$$

for $p = 1, \ldots, n-1$; $q = p+1, \ldots, n$; $r = 1, \ldots, n-1$ and $s = r+1, \ldots, n$. Thus, F is a matrix with dimension $\binom{n}{2} \times \binom{n}{2}$. The product defined by (2.29) has useful applications mainly in the theory of stability where stability properties of matrix eigenvalues or polynomial roots are examined, e.g. see [1, 4]. A well known result due to Stephanos [9] states that the eigenvalues of the matrix

$$\sum_{p,q=0}^{k} c_{pq} A^p \cdot A^q, \tag{2.30}$$

where k is a nonnegative integer, are given by

$$\frac{1}{2} \sum_{p,q=0}^{k} c_{pq} (\lambda_i^p \lambda_j^q + \lambda_j^p \lambda_i^q), \ 1 \leq i < j \leq n \tag{2.31}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. From (2.30), (2.31) and our definition of $s_{k,m}(\Lambda)$ in (2.2), it is easily seen that

$$\mathrm{tr} \sum_{p+q=k} A^p \cdot A^q = s_{k,2}(\Lambda), \ k = 0, 1, \ldots \tag{2.32}$$

and by equality (2.23), we have the following result.

Proposition 2.2 *Let $A \in M_n(\mathbb{C})$, $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A and s_k be given by (1.2). Then*

$$tr \sum_{p+q=k} A^p \cdot A^q = \frac{1}{2} \left(\sum_{p+q=k} s_p s_q - (k+1) s_k \right), \ k = 0, 1, \ldots. \tag{2.33}$$

If A in the above proposition is a nonnegative matrix then by (2.23) and Corollary 2.1, the trace in (2.33) satisfies

$$\mathrm{tr} \sum_{p+q=k} A^p \cdot A^q \geq \sum_{\mathbf{i} \in Q_{2,n}} h_k(a_{i_1 i_1}, a_{i_2 i_2}) \geq 0. \tag{2.34}$$

We note that the bialternate product generally does not preserve the nonnegativity of the multipliers, i.e. one can easily find matrices $A > 0$ and $B > 0$ such that $-A \cdot B > 0$. Also, by the definition of the bialternate product, it can be easily seen that

$$\sum_{\mathbf{i} \in Q_{2,n}} h_k(a_{i_1 i_1}, a_{i_2 i_2}) = \mathrm{tr} \sum_{p+q=k} (diag(A))^p \cdot (diag(A))^q. \tag{2.35}$$

Now, let $f(\lambda) \in \mathbb{C}[\lambda]$ be a polynomial with zeros $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and let $\mu_1, \ldots, \mu_{n-1}$ denote the zeros of $f^{(1)}(\lambda)$. By s_k and s'_k we denote the power sums of $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_{n-1}$, respectively, i.e. s_k is given by (1.2) and

$$s'_k = \sum_{i=1}^{n-1} \mu_i^k, \ k = 0, 1, \ldots. \tag{2.36}$$

Considering the case $m = 2$ in (2.7), we have

$$F_2(\lambda) = \frac{\lambda^2}{2} \frac{f^{(2)}(\lambda)}{f(\lambda)} = \frac{\lambda^2}{2} \frac{f^{(1)}(\lambda)}{f(\lambda)} \frac{f^{(2)}(\lambda)}{f^{(1)}(\lambda)}. \tag{2.37}$$

It is easily seen that for $\lambda > \max_{1\leq i\leq n} |\lambda_i|$, the last two fractions in (2.37) are expanded in power series with coefficients s_k and s'_k, respectively, so that after multiplication, it is obtained

$$F_2(\lambda) = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{p+q=k} s_p s'_q \lambda^{-k}. \tag{2.38}$$

From (2.38) and (2.9) with $m = 2$, we have

$$\sum_{p+q=k} s_p s'_q - 2s_{k,2}(\Lambda) = 0,\ \ k = 0, 1, \ldots \tag{2.39}$$

and by using (2.23), it follows that

$$\sum_{p+q=k} s_p(s'_q - s_q) + (k+1)s_k = 0,\ \ k = 0, 1, \ldots. \tag{2.40}$$

With $s_0 = n$ and $s'_0 = n - 1$, equality (2.40) is trivial and we shall omit this case. In the rest cases, (2.40) can be written in the following determinant form expressing s'_k as a polynomial of $s_1, \ldots, s_k$.

Proposition 2.3 *Given a polynomial $f(\lambda) \in \mathbb{C}[\lambda]$ of degree n,the power sums s_k of the zeros of $f(\lambda)$ and the power sums s'_k of the zeros of $f^{(1)}(\lambda)$ satisfy*

$$s'_k = s_k + \left(-\frac{1}{n}\right)^k \det\begin{bmatrix} s_1 & n & 0 & \ldots & 0 \\ 2s_2 & s_1 & n & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (k-1)s_{k-1} & s_{k-2} & s_{k-3} & \ldots & n \\ ks_k & s_{k-1} & s_{k-2} & \ldots & s_1 \end{bmatrix},\ \ k = 1, 2, \ldots. \tag{2.41}$$

Proof. Equality (2.41) can be written as

$$(-n)^k(s'_k - s_k) = D_k,\ \ k = 1, 2, \ldots \tag{2.42}$$

where D_k denotes the determinant in the right hand side of (2.41). Expanding this determinant by the entries of the last column gives

$$D_k = s_1 D_{k-1} - ns_2 D_{k-2} + n^2 s_3 D_{k-3} + \ldots + (-n)^{k-2}s_{k-1}D_1 + (-n)^{k-1}ks_k. \tag{2.43}$$

We shall use induction on k in order to prove equality (2.42). For $k = 1$ and $k = 2$, it can be easily seen that (2.42) follows from (2.40). Assume that (2.42) holds for determinants of dimension less than k, i.e.,

$$(-n)^i(s'_i - s_i) = D_i,\ \ i = 1, \ldots, k-1. \tag{2.44}$$

By substituting (2.44) in (2.43) and then (2.43) in (2.42), we obtain equality (2.40) with $s_0 = n$ and $s'_0 = n-1$. Thus, (2.42) holds for each $k = 1, 2, \ldots$, which completes the proof.

3 Concluding remarks

In the previous section, we defined a family of symmetric polynomials of the eigenvalues of a complex matrix A and found an explicit expression of each member of the family as a polynomial of the entries of A. This result is formulated in Theorem 2.1 and it includes the power sums $s_{k,1}(\Lambda)$ and the complete homogeneous symmetric polynomials $s_{k,n}(\Lambda)$ of matrix eigenvalues. In the latter case, we found a connection with MacMahon's Master Theorem and gave another form of representing $s_{k,n}(\Lambda)$ in terms of the entries of A. For $m = 2$, we have obtained $s_{k,2}(\Lambda)$ as a polynomial of the power sums $s_0, \ldots, s_k$ and this result is used in establishing equalities (2.33) and (2.41).

An immediate consequence of Theorem 2.1 is the family of inequalities (2.15) which are valid for the eigenvalues and diagonal entries of a nonnegative matrix. It should be noted that eigenvalue inequalities are of considerable interest in the theory of nonnegative matrices [2] and also in the study of some related matrix classes and problems. Typical examples in this respect provide the class of $M-$matrices [2] and the various forms of the inverse eigenvalue problem for nonnegative matrices [3]. In the latter case, it is clear that inequalities (2.15) give a necessary condition for an $n-$tuple of complex numbers $\lambda_1, \ldots, \lambda_n$ to be the spectrum of a nonnegative matrix with diagonal entries $a_{11}, \ldots, a_{nn}$. However, in the case of a nonnegative matrix A, it would be of some interest to further examine the properties of the sequence

$$s_{k,1}(\Lambda), s_{k,2}(\Lambda), \ldots, s_{k,n}(\Lambda) \tag{3.1}$$

where Λ is the spectrum of A and $k = 0, 1, \ldots$. A particular question about (3.1) is whether it is a unimodal sequence, i.e. whether there exists an index p such that $s_{k,1}(\Lambda) \leq s_{k,2}(\Lambda) \leq \ldots \leq s_{k,p}(\Lambda) \geq s_{k,p+1}(\Lambda) \geq \ldots \geq s_{k,n}(\Lambda)$. It is easily seen that this property is trivially satisfied for $k = 0$ and for $k = 1$, we have $s_{1,i}(\Lambda) = \binom{n-1}{i-1}(\lambda_1 + \ldots + \lambda_n)$, $i = 1, \ldots, n$ which is also a unimodal sequence.

We shall conclude the paper with the following

Conjecture 3.1 *Let $A \in M_n(\mathbb{R})$, $A \geq 0$, $f(\lambda) = \det(\lambda I - A)$ and μ_i, $i = 1, \ldots, n-1$ be the roots of $f^{(1)}(\lambda) = 0$. The power sums s'_k given by (2.36) satisfy*

$$s'_k \geq 0, \ k = 0, 1, \ldots. \tag{3.2}$$

The above conjecture is motivated by a previous work of the author [8] where it is shown that the zeros of each derivative of the characteristic polynomial of a nonnegative matrix A retain some of the basic spectral properties of A. On the other hand, inequalities (3.2) could be useful in studying the following inverse eigenvalue problem. Given an $n \times n$ nonnegative matrix with characteristic polynomial $f(\lambda)$, is there a nonnegative matrix with characteristic polynomial $\frac{\lambda^m}{n} f^{(1)}(\lambda)$ for some integer $m \geq 0$? Note that such a matrix can be easily found in the case $n = 3$. Finally, by taking into account (2.41), the above conjecture can be equivalently restated in terms of the power sums of the eigenvalues of A.

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