Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 63, No 9, 2010

MATHEMATIQUES

Mathematiques appliquées

ON THE PROPERTIES OF TWO MATRIX PRODUCTS ARISING IN STABILITY THEORY

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(*Submitted by Academician I. Popchev on April 8, 2010*)

Abstract

The paper is concerned with two special products of matrices which play a key role in the study of polynomial and matrix stability and in some problems arising in the theory of dynamical systems. The main results include a simple and elegant relation with the well-known Kronecker product and equalities involving traces of the product matrices.

Key words: bialternate product, Kronecker product, eigenvalues
2000 Mathematics Subject Classification: 15A15, 15A18, 15A69

1. Introduction. The notion of bialternate product of two square matrices is well known in matrix theory. Its original definition as "composition bialternée" together with some of its basic spectral properties appear in [1] and a later study with more contemporary results can be found in [2]. The major areas of application of this specific matrix product include polynomial and matrix stability analysis [2,3] and problems in the theory of dynamical systems associated with detecting and computing Hopf bifurcations in systems of ordinary differential equations [4,5]. In [6] another product of matrices is introduced and its properties and application in stability problems are studied in parallel with the bialternate product. The underlying idea in this reference is that both products of matrices can be analogously constructed by applying a well known principle of deriving a bilinear form from a given quadratic form. In the present paper, we re-define the matrix product from [6] by using the permanent function and within this framework, we present some new properties of the two matrix products.

This work was supported by project "Models of dynamical systems and information interactions" under Contract No 010099, Institute of Information Technologies.

2. Notation and definitions. For any integer $n \geq 2$, let $Q_{2,n}$ be the set of all pairs (i,j) such that $i,j \in \{1,2,\ldots,n\}$ and $i < j$, and let $G_{2,n}$ be the set of all pairs (i,j) such that $i,j \in \{1,2,\ldots,n\}$ and $i \leq j$. Clearly, $Q_{2,n}$ has $\binom{n}{2}$ elements and $G_{2,n}$ consists of $\binom{n+1}{2}$ elements. We shall assume that the elements of both $Q_{2,n}$ and $G_{2,n}$ are ordered lexicographically.

Given a matrix $A \in M_{m,n}(\mathbf{C})$, the notions of the k-th compound matrix $C_k(A)$ of A and the k-th induced matrix $P_k(A)$ of A for $k = 1,\ldots,r$, $r = \min\{m,n\}$ are well known in matrix theory [7]. For our purposes, only the second compound and the second induced matrices will be considered. We shall use double indices ij to label the rows and double indices kl to label the columns of these matrices where the pairs (i,j) and (k,l) are elements of given index sets. In particular, for $A = [a_{ij}] \in M_{m,n}(\mathbf{C})$, the entries of $C_2(A)$ are defined by

$$(1) \qquad C_2(A)_{ij,kl} = \det \begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}, \ (i,j) \in Q_{2,m}, \ (k,l) \in Q_{2,n}$$

and the entries of $P_2(A)$ are defined by

$$(2) \quad P_2(A)_{ij,kl} = \frac{1}{\sqrt{\mu(i,j)\mu(k,l)}} \operatorname{per} \begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}, \ (i,j) \in G_{2,m}, \ (k,l) \in G_{2,n},$$

where "per" denotes the permanent function and the value of $\mu(\cdot,\cdot)$ is determined as $\mu(p,q) = 2$ if $p = q$ and $\mu(p,q) = 1$ if $p \neq q$. Thus, the size of $C_2(A)$ is $\binom{m}{2} \times \binom{n}{2}$ and the size of $P_2(A)$ is $\binom{m+1}{2} \times \binom{n+1}{2}$. Compound and induced matrices have interesting spectral properties, e.g., see [7]. In particular, we recall that if λ_i, $1 \leq i \leq n$ are the eigenvalues of $A \in M_n(\mathbf{C})$ then the eigenvalues of $C_2(A)$ are $\lambda_i\lambda_j$, where $1 \leq i < j \leq n$ and the eigenvalues of $P_2(A)$ are $\lambda_i\lambda_j$ where $1 \leq i \leq j \leq n$.

Next, we give definitions of the two matrix products which are considered in the sequel. The first definition presents the well known bialternate product [1, 2].

Definition 1. *The bialternate product of matrices $A = [a_{ij}] \in M_n(\mathbf{C})$ and $B = [b_{ij}] \in M_n(\mathbf{C})$ is defined to be the matrix $F = A \cdot B$ where the entries of F are given by*

$$(3) \ f_{ij,kl} = \frac{1}{2}\left(\det \begin{bmatrix} a_{ik} & a_{il} \\ b_{jk} & b_{jl} \end{bmatrix} + \det \begin{bmatrix} b_{ik} & b_{il} \\ a_{jk} & a_{jl} \end{bmatrix}\right), \ (i,j) \in Q_{2,n}, \ (k,l) \in Q_{2,n}.$$

Using the permanent instead of determinant, we shall define another matrix product which will be referred to as a permanental bialternate product. It should be noted that our definition, although motivated by the work in [6], does not coincide with the definition given in this reference.

Definition 2. *The permanental bialternate product of matrices $A = [a_{ij}] \in M_n(\mathbf{C})$ and $B = [b_{ij}] \in M_n(\mathbf{C})$ is defined to be the matrix $G = A \times B$ where the entries of G are given by*

$$(4)\quad g_{ij,kl} = \frac{1}{2\sqrt{\mu(i,j)\mu(k,l)}}\left(\operatorname{per}\begin{bmatrix} a_{ik} & a_{il} \\ b_{jk} & b_{jl}\end{bmatrix} + \operatorname{per}\begin{bmatrix} b_{ik} & b_{il} \\ a_{jk} & a_{jl}\end{bmatrix}\right), \quad (i,j) \in G_{2,n},\ (k,l) \in G_{2,n}.$$

According to these definitions $F \in M_{\binom{n}{2}}(\mathbf{C})$, $G \in M_{\binom{n+1}{2}}(\mathbf{C})$ and it is seen from (1) and (2) that

$$(5)\qquad A \cdot A = C_2(A) \text{ and } A \times A = P_2(A).$$

Some basic algebraic properties of the two matrix products together with the main results of the paper are presented in the next section.

3. Main results. We begin with simple properties which are consequences of the definitions given by (3) and (4).

Proposition 1. *Let $A, B, C \in M_n(\mathbf{C})$. Then*

(a) $A \cdot B = B \cdot A$ *and* $A \times B = B \times A$;

(b) $(\alpha A) \cdot (\beta B) = \alpha\beta(A \cdot B)$ *and* $(\alpha A) \times (\beta B) = \alpha\beta(A \times B)$, *where* $\alpha, \beta \in \mathbf{C}$;

(c) $A \cdot (B + C) = A \cdot B + A \cdot C$ *and* $A \times (B + C) = A \times B + A \times C$;

(d) $(A \cdot B)^T = A^T \cdot B^T$ *and* $(A \times B)^T = A^T \times B^T$;

(e) $(A \cdot B)^* = A^* \cdot B^*$ *and* $(A \times B)^* = A^* \times B^*$, *where "$*$" denotes complex conjugate transpose;*

(f) $AB \cdot AB = (A \cdot A)(B \cdot B)$ *and* $AB \times AB = (A \times A)(B \times B)$;

(g) $(A \cdot A)^k = A^k \cdot A^k$ *and* $(A \times A)^k = A^k \times A^k$ *for* $k = 0, 1, 2, \ldots$;

(h) *If A is nonsingular then $A \cdot A$ and $A \times A$ are nonsingular matrices with* $(A \cdot A)^{-1} = A^{-1} \cdot A^{-1}$ *and* $(A \times A)^{-1} = A^{-1} \times A^{-1}$;

(i) *If A and B are upper triangular (lower triangular, diagonal) matrices with diagonal entries a_{ii} and b_{ii}, $i = 1, \ldots, n$, respectively, then $F = A \cdot B$ and $G = A \times B$ are upper triangular (lower triangular, diagonal) matrices with diagonal entries*

$$(6)\qquad f_{ij,ij} = \frac{1}{2}(a_{ii}b_{jj} + b_{ii}a_{jj}),\ (i,j) \in Q_{2,n}$$

and

$$(7)\qquad g_{ij,ij} = \frac{1}{2}(a_{ii}b_{jj} + b_{ii}a_{jj}),\ (i,j) \in G_{2,n},$$

respectively.

Proof. Conditions from (a) to (e) directly follow from (3) and (4). In view of (5), condition (f) essentially represents a well known property of the compound and induced matrices [7]. Condition (g) can be inductively obtained from (f) with $B = A$ and condition (h) follows from (f) with $B = A^{-1}$. To prove condition (i), we can first assume that A and B are upper triangular matrices. In this case, (6) and (7) simply follow from (3) and (4) by noting that the entries of A and B satisfy $a_{pq} = b_{pq} = 0$ for $1 \leq q \leq n-1$ and $p > q$. To see that F and G are upper triangular matrices as well, we have to show that the entries of F and G satisfy $f_{ij,kl} = 0$ and $g_{ij,kl} = 0$ for all pairs (i,j) and (k,l) such that $(i,j) > (k,l)$ in the lexicographical order of the sets $Q_{2,n}$ and $G_{2,n}$. Since $(i,j) > (k,l)$ if and only if either $i > k$ or $i = k$ and $j > l$, it is easily seen that equalities $f_{ij,kl} = 0$ and $g_{ij,kl} = 0$ are also obtained from (3) and (4) by taking into account the upper triangular structure of matrices A and B. The cases of lower triangular and diagonal matrices A and B follow in a similar way.

In what follows, we shall use the Kronecker product of matrices and some of its basic properties. The necessary background theory can be found in [8].

As usual, the Kronecker product is denoted by $\otimes$ and if $X \in M_{m,n}(\mathbf{C})$, $Y \in M_{p,q}(\mathbf{C})$ then $X \otimes Y \in M_{mp,nq}(\mathbf{C})$. Let $\{e_i : 1 \leq i \leq n\}$ be the standard basis of unit vectors in $\mathbf{C}^n$. By using the Kronecker product, the standard basis of unit vectors in $\mathbf{C}^{n^2}$ is given by $\{e_i \otimes e_j : 1 \leq i,j \leq n\}$. We shall denote by P the $n^2 \times n^2$ permutation matrix defined by equations

$$(8) \qquad P(e_i \otimes e_j) = e_j \otimes e_i, \ 1 \leq i,j \leq n.$$

Matrix P has interesting properties and useful applications, e.g., see [4,8,9]. It should be noted that $P = P^T = P^{-1}$, $P(x \otimes y) = y \otimes x$ for all $x, y \in \mathbf{C}^n$ and $P(A \otimes B) = (B \otimes A)P$ for all $A, B \in M_n(\mathbf{C})$. From (8), it is easily seen that P has an eigenvalue equal to -1 with algebraic and geometric multiplicity $\frac{1}{2}n(n-1)$, an eigenvalue equal to $+1$ with algebraic and geometric multiplicity $\frac{1}{2}n(n+1)$ and the eigenspaces corresponding to these eigenvalues are respectively given by

$$(9) \qquad E_{(-1)} = \operatorname{span}\{e_i \otimes e_j - e_j \otimes e_i : (i,j) \in Q_{2,n}\}$$

and

$$(10) \qquad E_{(+1)} = \operatorname{span}\{e_i \otimes e_j + e_j \otimes e_i : (i,j) \in G_{2,n}\}.$$

Subspaces (9) and (10) are orthogonal with respect to the usual inner product in $\mathbf{C}^{n^2}$, i.e., $\langle u, v\rangle = u^*v = 0$ for every $u \in E_{(-1)}$ and $v \in E_{(+1)}$. Also (9) and (10) are orthogonal complements in $\mathbf{C}^{n^2}$, i.e., $E_{(-1)} \cap E_{(+1)} = \{0\}$ and $E_{(-1)} \dot{+} E_{(+1)} = \mathbf{C}^{n^2}$, where $\dot{+}$ denotes the direct sum of subspaces. The next result shows that the pair of subspaces $(E_{(-1)}, E_{(+1)})$ is a reducing pair for $A \otimes B + B \otimes A$ where $A, B \in M_n(\mathbf{C})$.

Theorem 1. *Let $A = [a_{ij}] \in M_n(\mathbf{C})$ and $B = [b_{ij}] \in M_n(\mathbf{C})$. There is an orthogonal matrix $U \in M_{n^2}(\mathbf{R})$ such that*

$$U^T(A \otimes B + B \otimes A)U = 2\left[\begin{array}{c|c} A \cdot B & 0 \\ \hline 0 & A \times B \end{array}\right]. \tag{11}$$

Proof. Let V and W be matrices constructed as follows. The columns of V are given by

$$\frac{1}{\sqrt{2}}(e_i \otimes e_j - e_j \otimes e_i), \ (i,j) \in Q_{2,n}, \tag{12}$$

the columns of W are given by

$$\frac{1}{\sqrt{2\mu(i,j)}}(e_i \otimes e_j + e_j \otimes e_i), \ (i,j) \in G_{2,n}, \tag{13}$$

and the order of columns in V and W corresponds to the lexicographical order of the pairs (i,j) in $Q_{2,n}$ and $G_{2,n}$, respectively. It will be shown that the $n^2 \times n^2$ matrix $U = [V|W]$ satisfies (11).

It is easily seen that the columns of V and W provide orthogonal bases of $E_{(-1)}$ and $E_{(+1)}$, respectively, so that $U^TU = I$. Next, if $x \in E_{(-1)}$ then $Px = -x$ and since $P(A \otimes B) = (B \otimes A)P$, it follows that

$$P(A \otimes B + B \otimes A)x = -(A \otimes B + B \otimes A)x. \tag{14}$$

Similarly, if $x \in E_{(+1)}$ then $Px = x$ and it is obtained

$$P(A \otimes B + B \otimes A)x = (A \otimes B + B \otimes A)x. \tag{15}$$

Equalities (14) and (15) imply that $E_{(-1)}$ and $E_{(+1)}$ are invariant subspaces of $(A \otimes B + B \otimes A)$ and, since $V^TW = 0$ and $W^TV = 0$, we have

$$V^T(A \otimes B + B \otimes A)W = 0 \text{ and } W^T(A \otimes B + B \otimes A)V = 0. \tag{16}$$

Now, in order to prove (11) it remains to show that

$$V^T(A \otimes B + B \otimes A)V = 2A \cdot B \text{ and } W^T(A \otimes B + B \otimes A)W = 2A \times B. \tag{17}$$

For $(i,j) \in Q_{2,n}$ and $(k,l) \in Q_{2,n}$, the entry of $V^T(A \otimes B + B \otimes A)V$ in row ij and column kl is given by

$$\begin{aligned} & V^T(A \otimes B + B \otimes A)V_{ij,kl} = \\ = \ & \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)^T(A \otimes B + B \otimes A)(e_k \otimes e_l - e_l \otimes e_k) \\ = \ & \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)^T(Ae_k \otimes Be_l - Ae_l \otimes Be_k + Be_k \otimes Ae_l - Be_l \otimes Ae_k) \\ (18) = \ & a_{ik}b_{jl} - a_{il}b_{jk} + b_{ik}a_{jl} - b_{il}a_{jk}. \end{aligned}$$

In (18), we have used the mixed-product property of the Kronecker product and the fact that $(X \otimes Y)^T = X^T \otimes Y^T$. Analogously, for $(i,j) \in G_{2,n}$ and $(k,l) \in G_{2,n}$, the entry of $W^T(A \otimes B + B \otimes A)W$ in row ij and column kl is given by

$$\begin{aligned} & W^T(A \otimes B + B \otimes A)W_{ij,kl} = \\ & = \frac{1}{2\sqrt{\mu(i,j)\mu(k,l)}}(e_i \otimes e_j + e_j \otimes e_i)^T(A \otimes B + B \otimes A)(e_k \otimes e_l + e_l \otimes e_k) \\ (19) \quad & = \frac{1}{\sqrt{\mu(i,j)\mu(k,l)}}(a_{ik}b_{jl} + a_{il}b_{jk} + b_{ik}a_{jl} + b_{il}a_{jk}). \end{aligned}$$

By comparing (18) with (3) and (19) with (4), it follows that equalities (17) hold which completes the proof.

Theorem 1 shows that there is a simple and elegant relation between the matrix products defined by (3) and (4) and the Kronecker product of matrices.

Our next result gives a characterization of the traces of matrices $A \cdot B$ and $A \times B$. In particular, it is shown that the traces of $A \cdot B$ and $A \times B$ can be expressed in terms of the traces of A, B and AB.

Theorem 2. *Let $A = [a_{ij}] \in M_n(\mathbf{C})$ and $B = [b_{ij}] \in M_n(\mathbf{C})$. Then*

$$2\,\mathrm{tr}\,(A \cdot B) = \mathrm{tr}\,A\,\mathrm{tr}\,B - \mathrm{tr}\,(AB) \tag{20}$$

and

$$2\,\mathrm{tr}\,(A \times B) = \mathrm{tr}\,A\,\mathrm{tr}\,B + \mathrm{tr}\,(AB). \tag{21}$$

Proof. It follows from (3) that

$$\begin{aligned} 2\,\mathrm{tr}\,(A \cdot B) &= \sum_{(i,j) \in Q_{2,n}} f_{ij,ij} \\ &= \sum_{1 \le i < j \le n} (a_{ii}b_{jj} + b_{ii}a_{jj} - a_{ij}b_{ji} - b_{ij}a_{ji}) \\ (22) \quad &= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ii}b_{jj} - a_{ij}b_{ji}). \end{aligned}$$

Similarly, it is obtained from (4) that

$$\begin{aligned} 2\,\mathrm{tr}\,(A \times B) &= \sum_{(i,j) \in G_{2,n}} g_{ij,ij} \\ &= \sum_{1 \le i \le j \le n} \frac{1}{\mu(i,j)}(a_{ii}b_{jj} + b_{ii}a_{jj} + a_{ij}b_{ji} + b_{ij}a_{ji}) \\ (23) \quad &= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ii}b_{jj} + a_{ij}b_{ji}). \end{aligned}$$

By writing the right-hand sides of (22) and (23) in terms of the trace function, we obtain (20) and (21), respectively.

As immediate special cases of (20) and (21), we obtain

(24) $$2\,\mathrm{tr}\,(A \cdot I) = (n-1)\mathrm{tr}\,A \text{ and } 2\,\mathrm{tr}\,(A \times I) = (n+1)\mathrm{tr}\,A,$$

(25) $$2\,\mathrm{tr}\,C_2(A) = \mathrm{tr}^{\,2}A - \mathrm{tr}\,A^2 \text{ and } 2\,\mathrm{tr}\,P_2(A) = \mathrm{tr}^{\,2}A + \mathrm{tr}\,A^2.$$

Also, Theorem 2 together with Theorem 1 yield the trace equalities

(26) $$2(\mathrm{tr}\,A \cdot B + \mathrm{tr}\,A \times B) = \mathrm{tr}\,A\,\mathrm{tr}\,B = \mathrm{tr}\,(A \otimes B),$$

where the second equality represents a well known property of the Kronecker product of matrices, e.g., see [8].

4. Conclusion. The results in the previous section present specific properties of the bialternate and the permanental bialternate products of matrices. In particular, Theorem 1 establishes a relation of the two matrix products with the well known Kronecker product and Theorem 2 gives equalities expressing the traces of matrices $A \cdot B$ and $A \times B$ in terms of the traces of A, B and AB. In studying the properties of these products, it would be useful to find characterizations of other scalar-valued functions of $A \cdot B$ and $A \times B$, for example the determinant and permanent of these matrices. In this context, we notice the relation

(27) $$2^{n^2} \det(A \cdot B) \det(A \times B) = \det(A \otimes B + B \otimes A)$$

following from Theorem 1. Also, it would be of certain interest to extend the results in [10] for matrices $A \cdot B$ and $A \times B$, where A and B are assumed to be nonnegative matrices. Obviously, $A \times B$ is nonnegative in this case while $A \cdot B$ may have negative elements. Finally, we note that an examination of the structure and canonical forms of matrices $A \cdot B$ and $A \times B$ represents another topic of particular interest. Some results in this direction are obtained in [4].

REFERENCES

[1] Stéphanos C. Journal de Mathematiques Pures et Appliquees, **6**, 1900, Serie 5, 73–128.

[2] Fuller A. T. Journal of Mathematical Analysis and Applications, **23**, 1968, 71–98.

[3] Bellman R. Introduction to matrix analysis, New York, McGraw-Hill Book Company Inc., 1960.

[4] Govaerts W., B. Sijnave. Linear Algebra and Its Applications, **292**, 1999, Nos 1–3, 245–266.

[5] Guckenheimer J., M. Myers, B. Sturmfels. SIAM Journal of Numerical Analysis, **34**, 1997, No 1, 1–21.

[6] Elsner L. Unpublished manuscript, 1998.
[7] Marcus M., H. Minc. A survey of matrix theory and matrix inequalities, Rock Leigh New Jersey, Allyn and Bacon, 1964.
[8] Horn R. A., C. R. Johnson. Topics in matrix analysis, Cambridge, Cambridge University Press, 1995.
[9] Elsner L., V. Monov, T. Szulc. Linear and Multilinear Algebra, **50**, 2002, No 3, 199–218.
[10] Monov V. Compt. rend. Acad. bulg. Sci., **63**, 2010, No 2, 189–196.

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