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MATHEMATIQUES

Mathematiques appliquées

ON THE PROPERTIES OF TWO MATRIX PRODUCTS ARISING IN STABILITY THEORY

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Abstract

The paper is concerned with two special products of matrices which play a key role in the study of polynomial and matrix stability and in some problems arising in the theory of dynamical systems. The main results include a simple and elegant relation with the well-known Kronecker product and equalities involving traces of the product matrices.

Key words: bialternate product, Kronecker product, eigenvalues
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1. Introduction. The notion of bialternate product of two square matrices is well known in matrix theory. Its original definition as "composition bialternée" together with some of its basic spectral properties appear in [1] and a later study with more contemporary results can be found in [2]. The major areas of application of this specific matrix product include polynomial and matrix stability analysis [2,3] and problems in the theory of dynamical systems associated with detecting and computing Hopf bifurcations in systems of ordinary differential equations [4,5]. In [6] another product of matrices is introduced and its properties and application in stability problems are studied in parallel with the bialternate product. The underlying idea in this reference is that both products of matrices can be analogously constructed by applying a well known principle of deriving a bilinear form from a given quadratic form. In the present paper, we re-define the matrix product from [6] by using the permanent function and within this framework, we present some new properties of the two matrix products.

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2. Notation and definitions. For any integer $n \geq 2$, let $Q_{2,n}$ be the set of all pairs (i,j) such that $i, j \in \{1, 2, \dots, n\}$ and $i < j$, and let $G_{2,n}$ be the set of all pairs (i,j) such that $i, j \in \{1, 2, \dots, n\}$ and $i \leq j$. Clearly, $Q_{2,n}$ has $\binom{n}{2}$ elements and $G_{2,n}$ consists of $\binom{n+1}{2}$ elements. We shall assume that the elements of both $Q_{2,n}$ and $G_{2,n}$ are ordered lexicographically.

Given a matrix $A \in M_{m,n}(\mathbf{C})$, the notions of the k-th compound matrix $C_k(A)$ of A and the k-th induced matrix $P_k(A)$ of A for $k = 1, \dots, r$, $r = \min\{m, n\}$ are well known in matrix theory [7]. For our purposes, only the second compound and the second induced matrices will be considered. We shall use double indices ij to label the rows and double indices kl to label the columns of these matrices where the pairs (i,j) and (k,l) are elements of given index sets. In particular, for $A = [a_{ij}] \in M_{m,n}(\mathbf{C})$, the entries of $C_2(A)$ are defined by

$$C_2(A)_{ij,kl} = \det \begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}, \ (i,j) \in Q_{2,m}, \ (k,l) \in Q_{2,n} \tag{1}$$

and the entries of $P_2(A)$ are defined by

$$P_2(A)_{ij,kl} = \frac{1}{\sqrt{\mu(i,j)\mu(k,l)}} \operatorname{per} \begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}, \ (i,j) \in G_{2,m}, \ (k,l) \in G_{2,n}, \tag{2}$$

where "per" denotes the permanent function and the value of $\mu(\cdot,\cdot)$ is determined as $\mu(p,q) = 2$ if $p = q$ and $\mu(p,q) = 1$ if $p \neq q$. Thus, the size of $C_2(A)$ is $\binom{m}{2} \times \binom{n}{2}$ and the size of $P_2(A)$ is $\binom{m+1}{2} \times \binom{n+1}{2}$. Compound and induced matrices have interesting spectral properties, e.g., see [7]. In particular, we recall that if λ_i, $1 \leq i \leq n$ are the eigenvalues of $A \in M_n(\mathbf{C})$ then the eigenvalues of $C_2(A)$ are $\lambda_i \lambda_j$, where $1 \leq i < j \leq n$ and the eigenvalues of $P_2(A)$ are $\lambda_i \lambda_j$ where $1 \leq i \leq j \leq n$.

Next, we give definitions of the two matrix products which are considered in the sequel. The first definition presents the well known bialternate product [1, 2].

Definition 1. *The bialternate product of matrices $A = [a_{ij}] \in M_n(\mathbf{C})$ and $B = [b_{ij}] \in M_n(\mathbf{C})$ is defined to be the matrix $F = A \cdot B$ where the entries of F are given by*

$$f_{ij,kl} = \frac{1}{2} \left(\det \begin{bmatrix} a_{ik} & a_{il} \\ b_{jk} & b_{jl} \end{bmatrix} + \det \begin{bmatrix} b_{ik} & b_{il} \\ a_{jk} & a_{jl} \end{bmatrix} \right), \ (i,j) \in Q_{2,n}, \ (k,l) \in Q_{2,n}. \tag{3}$$

Using the permanent instead of determinant, we shall define another matrix product which will be referred to as a permanental bialternate product. It should be noted that our definition, although motivated by the work in [6], does not coincide with the definition given in this reference.