FEM Nonlinear Vibration Analysis of Plates

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Abstract. Plates are structures which have wide applications among engineering constructions. The knowledge of the dynamical behavior of plates is important for their design and maintenance. The dynamical behavior of the plate can change significantly due to bifurcation points which arise due to the nonlinear terms at the equation of motion in the presence of large displacements.

The current work presents numerical methods for investigating the dynamical behavior of plates with complex geometry and studies the scalability and efficiency of the methods. The potential of the methods is demonstrated on a plate with complex geometry. Its bifurcation diagrams are computed, stability of the solution is determined, secondary branches are obtained and the corresponding shapes of vibration are shown.

1 Introduction

Plates are thin structures and due to strong external loads their displacements can become large. Linear theories are not appropriate for modeling large displacements thus one should include geometrical nonlinear terms at the equation of motion for obtaining more accurate results. Nonlinearities can change drastically the behavior of the system, thus additional tools for analyzing such systems need to be used. The aim of the work is to present efficient numerical methods, suitable for parallel implementation, for analyzing nonlinear dynamical systems which arise from motion of plates with complex geometry.

The equation of motion of the plate is derived by the finite element method assuming classical plate theory and including geometrically nonlinear terms. Triangular finite elements are used, thus one can model plates of complex shapes (Fig. 1). The shooting method is used to compute the periodic responses of the plate due to harmonic excitations. Prediction for the next point from the bifurcation diagram is defined by the continuation method. Stability is determined by the Floquet's multipliers. Bifurcation points are found, the corresponding secondary branches and the associated shapes of vibration are presented. The numerical methods are run on parallel processes and their scalability is studied.

2 Equation of motion of plates

The nonlinear equation of motion of plate is derived in Cartesian coordinate system assuming classical plate theory, also known as Kirchoff's hypotheses. Only transverse displacements are considered on the middle plane. Kirchoff's hypotheses states that stresses in the direction normal to the plate middle surface are negligible and strains vary linearly within the plate thickness.

Assuming Kirchoff's hypotheses, the in-plane displacements $u(x,y,z,t)$ and $v(x,y,z,t)$ and the out-of-plane displacement $w(x,y,z,t)$ are expressed by the out-of-plane displacement on the middle plane $w_0(x,y,t)$:

$$\begin{aligned} u(x,y,z,t) &= -z\frac{\partial w_0(x,y,t)}{\partial x}, \\ v(x,y,z,t) &= -z\frac{\partial w_0(x,y,t)}{\partial y}, \\ w(x,y,z,t) &= w_0(x,y,t), \end{aligned} \tag{1}$$

The middle plane is defined for $z=0$. Using the nonlinear strain-displacement relations from Green's strain tensor and assuming that ε_z, γ_{xz} and γ_{yz} are negligible, i.e. $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$, the following expressions are obtained:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = -z\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2, \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = -z\frac{\partial^2 w_0}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2, \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = -2z\frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}. \end{aligned} \tag{2}$$

The stresses are related to the strains by the constitutive relations written in reduced form. For isotropic materials this relation is given by:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \tag{3}$$

where E is Young's modulus and ν is Poisson's ratio. The equation of motion is derived by the Hamilton principle:

$$\int_{t_1}^{t_2} (\delta T - \delta \mathit{\Pi}) dt = 0, \tag{4}$$

where δT and $\delta \mathit{\Pi}$ are variations of the kinetic and potential energies:

$$\mathit{\Pi} = \int_V (\varepsilon_x \sigma_x + \varepsilon_y \sigma_y + \gamma_{xy}\tau_{xy}) dV \tag{5}$$

$$T = \rho \int_V (u\ddot{u} + v\ddot{v} + w\ddot{w}) dV, \tag{6}$$

where by double dot is denoted the second derivative with respect to time, V is the volume of the plate and ρ is the density. The equation of motion is obtained in the following form:

$$\begin{gathered}
\rho h \frac{\partial^2 w_0}{\partial t^2} + \frac{Eh^3}{12(1-\nu^2)} \left(\frac{\partial^4 w_0}{\partial x^4} + 2\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{\partial^4 w_0}{\partial y^4} \right) = \\
= \frac{h^3}{12} \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} \right) + N_x \frac{\partial^2 w_0}{\partial x^2} + 2N_{xy} \frac{\partial^2 w_0}{\partial x \partial y} + N_y \frac{\partial^2 w_0}{\partial y^2}
\end{gathered} \tag{7}$$

where h is the thickness and N_x, N_y and N_{xy} are the stress resultants given by:

$$\begin{aligned}
N_x &= \frac{Eh}{1-\nu^2} \left(\frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 + \frac{\nu}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 \right), \\
N_y &= \frac{Eh}{1-\nu^2} \left(\frac{1}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 + \frac{\nu}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right), \\
N_{xy} &= \frac{Eh}{2(1+\nu)} \left(\frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \right).
\end{aligned} \tag{8}$$

Details about the derivation of the equation of motion can be found, for example in [1].

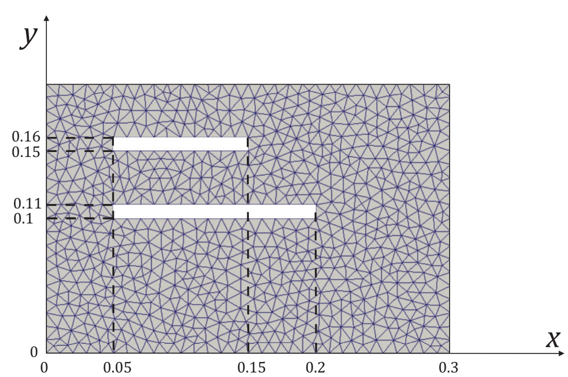

Fig. 1. Plate with complex geometry and mesh of finite elements.

3 Finite element method and computation of periodic responses

The partial differential equation (7) is discretized by the finite element method. A system of nonlinear ordinary differential equations of the following type is obtained:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K_L}\mathbf{q}(t) + \mathbf{K_{NL}}(\mathbf{q}(t))\mathbf{q}(t) = \mathbf{F}(t) \tag{9}$$

Due to the complexity of the geometry of the plate and the necessity of using fine mesh of elements for convergence, the resulting system (9) can become large. Efficient parallel algorithms for solving large systems of sparse and dense matrices are used in the analysis.

Dynamical analysis is performed by investigating the periodic responses of the plate due to harmonic external forces. Of interest is to determine how the response changes with change of the excitation frequency. Periodic responses are computed by the shooting method. It computes iteratively the initial conditions which lead to periodic response. Shooting method consists of time integration of $2N$ independent systems, where N is the total number of degrees of freedom, and it is very suitable for parallel implementation [2]. Prediction for the next point from the bifurcation diagram is defined by the continuation method. It is shown that the forced periodic responses of plates with complex geometries present motion which combines several modes of vibration (Fig. 2)

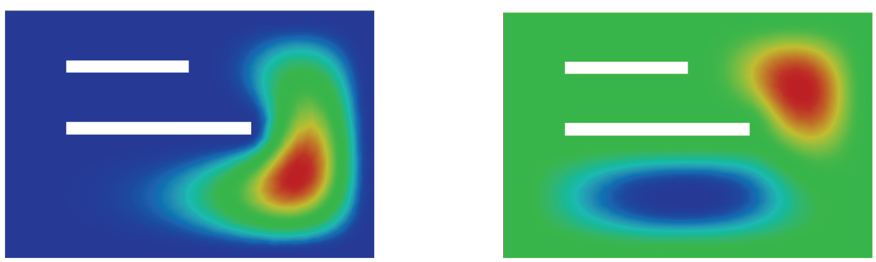

Fig. 2. First and second linear modes of the plate from Fig. 1.

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