Robust Multigrid for Isogeometric Analysis

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1 Isogeometric model problem

For the sake of simplicity, we restrict ourselves to the following model problem. Let $\Omega = (0,1)^d$ and assume $f \in L^2(\Omega)$ to be a given function. Find a function $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

In variational form, this problem reads: find $u \in H^1(\Omega)$ such that

$$a(u,v) := (\nabla u, \nabla v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)} \qquad \forall v \in H^1(\Omega).$$

We obtain an isogeometric discretization of this problem by choosing a sequence of spline spaces $\mathcal{V}_\ell \subset H^1(\Omega)$, $\ell = 0,1,\ldots$, and introducing the Galerkin discretization: find $u_\ell \in \mathcal{V}_\ell$ such that

$$a(u_\ell, v_\ell) = \langle f, v_\ell \rangle \qquad \forall v_\ell \in \mathcal{V}_\ell.$$

In the 1D setting, $\mathcal{V}_\ell$ is chosen as a spline space of some fixed degree p over a uniform, open knot vector consisting of $n_\ell = n_0 2^\ell$ subintervals of length $h_\ell := \frac{1}{n_\ell} = \frac{1}{n_0} 2^{-\ell}$ by uniform dyadic refinement. "Open" here refers to the fact that the first and last knots are repeated $p+1$ times. All interior knots are simple and thus the spline space has the maximum continuity, $\mathcal{V}_\ell \subset C^{p-1}(0,1)$. Furthermore, the spaces are nested, i.e., $\mathcal{V}_\ell \subset \mathcal{V}_{\ell+1}$. If we want to make explicit mention of the spline degree, we write $S_{p,\ell} = \mathcal{V}_\ell$ for this uniformly refined spline space.

In higher dimensions, the space $\mathcal{V}_\ell$ is taken as the tensor product of 1D spline spaces as just described. Whenever a basis for these spline spaces is needed, we use the canonical basis of normalized B-splines or tensor products thereof. For more details on splines, see, e.g., [1].

2 Description of the multigrid algorithm

Denoting the stiffness matrix on level ℓ by K_ℓ, the multigrid algorithm for solving the discretized equation on grid level ℓ reads as follows. Starting from an initial approximation $\underline{u}_\ell^{(0)}$, one iteration of the multigrid method to obtain the next iterate $\underline{u}_\ell^{(1)}$ is given by the following two steps:

- *Smoothing procedure:* For some fixed number ν of smoothing steps, compute

$$\underline{u}_\ell^{(0,m)} := \underline{u}_\ell^{(0,m-1)} + \tau L_\ell^{-1} \left(\underline{f}_\ell - K_\ell \, \underline{u}_\ell^{(0,m-1)} \right) \qquad \text{for } m = 1, \ldots, \nu, \quad (1)$$

where $\underline{u}_\ell^{(0,0)} := \underline{u}_\ell^{(0)}$. The choice of the smoothing matrix L_ℓ^{-1} and the damping parameter $\tau > 0$ will be discussed below.

- *Coarse-grid correction:*

  - Compute the defect and restrict it to grid level $\ell-1$ using a restriction matrix $I_\ell^{\ell-1}$:

$$\underline{r}_{\ell-1}^{(1)} := I_\ell^{\ell-1} \left(\underline{f}_\ell - K_\ell \, \underline{u}_\ell^{(0,\nu)} \right).$$

  - Compute the update $\underline{p}_{\ell-1}^{(1)}$ by solving the coarse-grid problem

$$K_{\ell-1} \, \underline{p}_{\ell-1}^{(1)} = \underline{r}_{\ell-1}^{(1)}. \quad (2)$$

  - Prolongate $\underline{p}_{\ell-1}^{(1)}$ to the grid level ℓ using a prolongation matrix $I_{\ell-1}^\ell$ and add the result to the previous iterate:

$$\underline{u}_\ell^{(1)} := \underline{u}_\ell^{(0,\nu)} + I_{\ell-1}^\ell \, \underline{p}_{\ell-1}^{(1)}.$$

We denote by $T_\ell = I - I_{\ell-1}^\ell \, K_{\ell-1}^{-1} \, I_\ell^{\ell-1} K_\ell$ the action of the coarse-grid correction.

3 A robust smoother for IGA

Lemma 1. *If the approximation property*

$$\|T_\ell \underline{v}\|_{L_\ell} \le c \|\underline{v}\|_{K_\ell} \qquad \forall \underline{v} \in \mathbb{R}^{m_\ell},$$

and the smoothing property

$$K_\ell \le c L_\ell$$

are satisfied with uniform constants c which do not depend on ℓ or p, then the two-grid algorithm converges with a rate which is robust in ℓ and p.

The construction of our smoother depends on first showing that the mass matrix is a robust smoother in a large subspace of the spline space, and then extending this smoother to the whole space by a low-rank correction.

In [2] it was shown that a robust inverse estimate holds for the following large subspace of $S_{p,\ell}$.

Definition 1. *We denote by $\widetilde{S}_{p,\ell}$ the space of all $u_\ell \in S_{p,\ell}$ whose odd derivatives of order less than p vanish at the boundary.*

Theorem 1 ([2]). *Let $\ell \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then*

$$|u_\ell|_{H^1(0,1)} \leq 2\sqrt{3} h_\ell^{-1} \|u_\ell\|_{L^2(0,1)} \qquad \forall u_\ell \in \widetilde{S}_{p,\ell}.$$

Furthermore an approximation property holds in $\widetilde{S}_{p,\ell}$. Below, $\widetilde{\Pi}_\ell$ denotes a suitably chosen orthogonal projector into $\widetilde{S}_{p,\ell}$.

Theorem 2. *Let $\ell \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then*

$$\|u - \widetilde{\Pi}_\ell u\|_{L^2(0,1)} \leq 2\sqrt{2} h_\ell |u|_{H^1(0,1)} \qquad \forall u \in H^1(0,1).$$

The above results allow us to prove the assumptions of Lemma 1 for the smoother $L_\ell := h_\ell^{-2} M_\ell$ in the subspace $\widetilde{S}_{p,\ell}$. The following abstract result allows us the extension to the entire space $S_{p,\ell}$. We drop subscripts here to emphasize the abstract nature of the result.

Lemma 2. *Assume that the smoother L satisfies the properties*

$$\begin{aligned} \|Tv\|_L &\leq c\|v\|_K && \forall v \in S, \\ \|\tilde{v}\|_K &\leq c\|\tilde{v}\|_L && \forall \tilde{v} \in \widetilde{S}, \\ \|(I - \widetilde{\Pi})v\|_L &\leq c\|v\|_K && \forall v \in S. \end{aligned}$$

Then the modified smoother $\widehat{L} := L + (I - \widetilde{\Pi})^T K (I - \widetilde{\Pi})$ satisfies the assumptions of Lemma 1 robustly.

A slight generalization allows us the construction of a robust smoother for the 1D case in the form

$$\widehat{L}_\ell := h_\ell^{-2} M_\ell + \widetilde{K}_\ell := h_\ell^{-2} M_\ell + (I - \Pi_\ell^I)^T K (I - \widetilde{\Pi}_\ell^I),$$

where Π_ℓ^I is now a suitably chosen orthogonal projector onto the space of "inner" splines, that is, discarding the p left- and right-most B-spline basis functions. The term $\widetilde{K}_\ell$ is then nothing but a Schur complement. In 2D, the smoother

$$h_\ell^2 \widehat{L}_\ell \otimes \widehat{L}_\ell - h_\ell^2 \widetilde{K}_\ell \otimes \widetilde{K}_\ell$$

can be shown to be robust by relying on the 1D results. The second term is a low-rank correction and the smoother can be efficiently realized by means of the Sherman-Morrison-Woodbury formula.

4 Experimental results

As a numerical example, we solve the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega$$

on the domain $\Omega = (0,1)^d$, $d = 1, 2$, where the right-hand side and boundary conditions are chosen in accordance with the exact solution $u(\mathbf{x}) = \prod_{j=1}^d \sin(\pi x_j)$.

We perform a (tensor product) B-spline discretization using uniformly sized knot spans and maximum-continuity splines for varying spline degrees p. We start from a coarse discretization with only a single interval and perform ℓ uniform, dyadic refinement steps to obtain a finer discretization.

We then set up a two-grid method as previously described with the proposed smoothers and one pre- and post-smoothing steps, respectively.

We perform two-grid iteration until the Euclidean norm of the initial residual is reduced by a factor of 10^{-8}. The iteration numbers using different spline degrees p as well as different refinement levels ℓ for the one-dimensional domain are given in Table 1, and those for the two-dimensional domain in Table 2. As predicted by the theory, the iteration numbers remain uniformly bounded with respect to the spline degree p as well as the refinement level.

p	1	2	3	4	6	8	10	12	14	16	18	20
$\ell = 10$	22	20	20	21	20	20	20	20	18	18	17	17
$\ell = 11$	23	20	20	21	20	20	19	19	18	19	18	17
$\ell = 12$	23	20	20	20	20	20	20	19	18	18	18	18

Table 1: Two-grid iteration numbers in 1D.

p	2	3	4	6	8	10	11	12	13	14	15	16
$\ell = 5$	82	80	75	76	72	70	71	70	68	69	69	66
$\ell = 6$	83	87	76	75	72	70	70	69	68	68	67	65

Table 2: Two-grid iteration numbers in 2D.

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