Building of Numerically Effective Kalman Estimator Algorithm for Urban Transportation Network *

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Abstract. The well known store-and-forward model of urban network is concerned. An effective numerical algorithm of Kalman observer is represented. The proposed approach is based on resolvent method and use special structure of the model and relation between discrete and continuous type algebraic Riccati equations. In a final part of the paper a numerical performance was evaluated to show an efficiency of the proposed approach.

Keywords: Kalman observer, algebraic Riccati equation, resolvent method, urban network.

1 Problem formulation

An urban transportation system is the network of intersections which are controlled by the traffic signals. The well known Gazis and Potts store-and-forward model [1], [2] of the traffic network was studied in the paper. In the oversaturated urban network a coming to intersections vehicles generate *n* queues with x_i vehicles in it. Let low subscript $i \in \{1, 2, ..., n\}$ indicates any variable that is related to the *i*-th queue. Figure 1 illustrates all flows related to *i*-th queue.

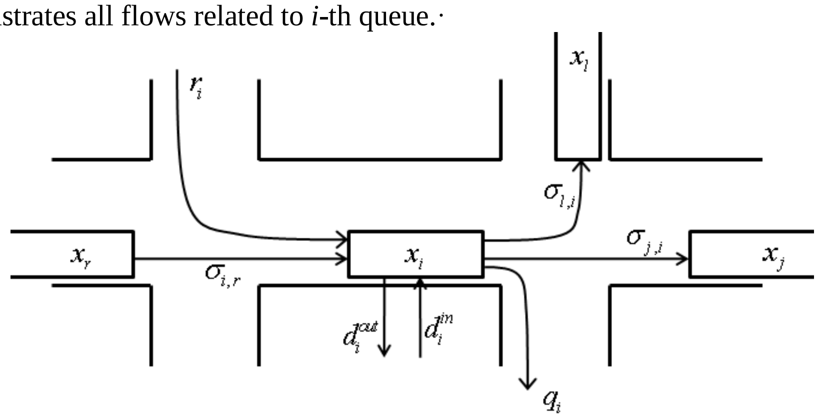

Fig. 1. The flows related with *i*-th queue

On figure 1 are denoted: r_i and q_i are the external network's inflow and outflow respectively; d_i^{in} and d_i^{out} are the exit flow within *i*-th link respectively; any $\sigma_{i,j}$ denote exchange flow from *j*-th to *i*-th queue.

Traffic signals change periodically their phases during one cycle period c to provide each queue right of way. Each phase corresponding to the *i*-th queue comprises a loss time l_i and an effective green time

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adfa, p. 1, 2011.

g_i. Then, during traffic signals control period $\left[kT,\ (k+1)T\right]$, a road network may be represented as a discrete-time linear state-space model

$$\left.\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k) + \mathbf{F}_d\mathbf{w}(k),\ \mathbf{x}(0)=\mathbf{x}_0;\\ \mathbf{y}(k) &= \mathbf{C}_d\mathbf{x}(k) + \mathbf{v}(k),\end{aligned}\right\} \tag{1}$$

where T is the discrete time step (as usual equal to cycle period c); $k=0,1,...$ is the discrete-time index; $\mathbf{x}(k)=\left[x_1,\ x_2,\ ...,\ x_n\right]^T \in R^n$ is the state vector; $\mathbf{u}(k)\in R^m$ and $\mathbf{w}(k)=\left[w_1,\ w_2,\ ...,\ w_n\right]^T \in R^n$ are the control and disturbances vectors respectively, are the deviations from their nominal values; m is a number of $g_j^e,\ j\in\{1,2,...,m\},\ m<n$ independent effective green times g_i ; $\mathbf{y}(k)$, $\mathbf{v}(k)=\left[v_1,\ v_2,\ ...,\ v_n\right]^T \in R^n$ are measurement and measurement noise vectors respectively; finally model (1) constant matrices can be written as

$$\mathbf{A}_d=\mathbf{I}_n,\ \mathbf{B}_d\in R^{n\times m},\ \mathbf{F}_d=diag\left(\left[f_1,\ f_2,\ ...,\ f_n\right]\right)\in R^{n\times n},\ \mathbf{C}_d=\mathbf{I}_n\,, \tag{2}$$

where $\mathbf{I}_n$ is a $n\times n$ identity matrix and $\mathbf{B}_d$ is a matrix containing the network characteristics (network topology, saturation flows, average turning rates).

For the system (1) the processes $\mathbf{w}(k)$, $\mathbf{v}(k)$ may be supposed as Gaussian centered stationary white noises with covariance matrices $E\left\{\mathbf{w}(k)\mathbf{w}^T(k)\right\}=\mathbf{Q}_d\geq 0$, $E\left\{\mathbf{v}(k)\mathbf{v}^T(k)\right\}=\mathbf{R}_d>0$ respectively.

Obviously processes v_i are independent to each other and appropriate covariance matrix can be written as $\mathbf{R}_d=diag\left(\left[r_1^d,\ r_2^d,\ ...,\ r_n^d\right]\right)$. Then, as it was mentioned matrix $\mathbf{B}_d$ contain average turning rates. So if in the actual flow the passed vehicles were divided not proportionally to the mentioned rates then overflow of one direction means underflow of another one direction, i.e. processes w_i are correlated to each other. Due to this fact the covariance matrix $\mathbf{Q}_d$ is a symmetric matrix.

An effective control [2] schemes of the urban traffic needs the estimation of the queue lengths despite of external disturbances and measurement noises. The optimal Kalman estimator [4] construct a state estimate $\hat{\mathbf{x}}(k)$ that minimizes the mean square error

$$E\left\{\left(\hat{\mathbf{x}}(k)-\mathbf{x}(k)\right)\left(\hat{\mathbf{x}}(k)-\mathbf{x}(k)\right)^{\mathrm{T}}\right\}\to\min \tag{3}$$

and is represented as the following state equation

$$\hat{\mathbf{x}}(k+1)= \mathbf{A}_d\hat{\mathbf{x}}(k) + \mathbf{B}_d\mathbf{u}(k) + \mathbf{L}_d\left(\mathbf{y}(k)-\mathbf{C}_d\hat{\mathbf{x}}(k)\right),\ \hat{\mathbf{x}}(0)=\hat{\mathbf{x}}_0\,, \tag{4}$$

where estimator gain matrix $\mathbf{L}_d$ is given by $\mathbf{L}_d=\mathbf{A}_d\mathbf{X}\mathbf{C}_d^T(\mathbf{C}_d\mathbf{X}\mathbf{C}_d^T+\mathbf{R}_d)^{-1}$;
$\mathbf{X}$ is derived by solving a discrete algebraic Riccati equation (DARE)

$$\mathbf{X}=\mathbf{A}_d\mathbf{X}\mathbf{A}_d^T+\mathbf{F}_d\mathbf{Q}_d\mathbf{F}_d^T-\mathbf{A}_d\mathbf{X}\mathbf{C}_d^T(\mathbf{C}_d\mathbf{X}\mathbf{C}_d^T+\mathbf{R}_d)^{-1}\mathbf{C}_d\mathbf{X}\mathbf{A}_d^T\,. \tag{5}$$

The paper studies question of building Kalman estimator (4) for urban transportation network. As it is seen the problem (3) solution lies in an effective numerical algorithm for DARE (5) with respect of special structure of matrices (2). The paper proposes solution of the problem by resolvent method.

2 Building of discrete Kalman estimator for urban transportation network by resolvent method

2.1 Development of resolvent method

It is known [3] that there is a relation between DARE and continuous type algebraic Riccati equation (CARE)

$$\mathbf{A}^T\mathbf{X}+\mathbf{XA}\ \ +\mathbf{P}-\mathbf{XQX}=\mathbf{0}_{n,n}\,. \tag{6}$$

This relation allow us to use numerical algorithm to solve DARE by solving CARE with known matrices of (6) expressed as

$$\left.\begin{aligned}&\mathbf{A}=\mathbf{I}_n-2(\mathbf{\Delta}^{-1})^T;\\&\mathbf{Q}=2(\mathbf{I}_n+\mathbf{A}_d^T)^{-1}\mathbf{C}_d^T\mathbf{R}_d^{-1}\mathbf{C}_d\mathbf{\Delta}^{-1};\\&\mathbf{P}=2\mathbf{\Delta}^{-1}\mathbf{F}_d\mathbf{Q}_d\mathbf{F}_d^T(\mathbf{I}_n+\mathbf{A}_d^T)^{-1};\\&\mathbf{\Delta}=\mathbf{I}_n+\mathbf{A}_d+\mathbf{F}_d\mathbf{Q}_d\mathbf{F}_d^T(\mathbf{I}_n+\mathbf{A}_d^T)^{-1}\mathbf{C}_d^T\mathbf{R}_d^{-1}\mathbf{C}_d\,.\end{aligned}\right\} \tag{7}$$

Stabilizing solution of CARE can be found by resolvent method [4] from linear equation

$$\mathbf{U}_2\mathbf{X}+\mathbf{U}_1=\mathbf{0}_{2n,n}\,,\ \mathbf{U}_1,\,\mathbf{U}_2\in R^{2n\times n}\,, \tag{8}$$

where $2n\times 2n$ real matrix $\mathbf{U}=\begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{bmatrix}$ is defined as integral $\mathbf{U}=\dfrac{1}{2\pi j}\displaystyle\int_C\mathbf{\Theta}(s)ds$ from resolvent $\mathbf{\Theta}(s)=(s\mathbf{I}_{2n}-\mathbf{H})^{-1}$ of Hamilton matrix $\mathbf{H}=\begin{bmatrix}\mathbf{A} & -\mathbf{Q}\\ -\mathbf{P} & -\mathbf{A}^T\end{bmatrix}$.

In the paper for matrix $\mathbf{U}$ has been found a representation

$$\mathbf{U}=0.5\mathbf{I}_{2n}+\mathbf{HS}\,, \tag{9}$$

where

$$\mathbf{S}=\sum_{k=0}^{n-1}\eta_{n-1-k}\mathbf{G}^k\ ,\ \mathbf{G}=\mathbf{H}^2\,; \tag{10}$$

$$\eta_m=\gamma_0\delta_m+\gamma_1\delta_{m-1}+...+\gamma_m\delta_0\,,\ \delta_0=1\,,\ m=0,1,...,n-1\,; \tag{11}$$

$\delta_1,\,\delta_2,...,\delta_{n-1}$ are coefficients of polynomial

$$\delta(x)=x^n+\delta_1x^{n-1}+...+\delta_{n-1}x+\delta_n\,; \tag{12}$$

$$\gamma_r=\frac{1}{2\pi j}\int_C\frac{s^{2r}}{\delta(x)}ds,\ x=s^2\,,\ r=0,1,...,n-1 \tag{13}$$

are values of quadratic functionals, which, as it shown in the paper, can be easily computed as *n* definite integrals.

Then, for $2n\times 2n$ sizes matrices $\mathbf{G}=\mathbf{H}^2$, $\mathbf{S}$, $\mathbf{U}$, $\mathbf{G}^k$, $k=1,2,...$ were considered block representation with $n\times n$ size matrices

$$\mathbf{G}=\begin{bmatrix}\mathbf{A}^2+\mathbf{QP} & \mathbf{QA}^T-\mathbf{AQ}\\ \mathbf{A}^T\mathbf{P}-\mathbf{PA} & \mathbf{PQ}+\mathbf{A}^{2T}\end{bmatrix}=\begin{bmatrix}\mathbf{G}_{11} & \mathbf{G}_{12}\\ \mathbf{G}_{21} & \mathbf{G}_{22}\end{bmatrix},\ \mathbf{G}^k=\begin{bmatrix}\mathbf{G}_k^{11} & \mathbf{G}_k^{12}\\ \mathbf{G}_k^{21} & \mathbf{G}_k^{22}\end{bmatrix},\ k=1,2,...\,; \tag{14}$$

$$\mathbf{S}=\begin{bmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{21} & \mathbf{S}_{22}\end{bmatrix},\mathbf{U}=\begin{bmatrix}\mathbf{U}_{11} & \mathbf{U}_{12}\\ \mathbf{U}_{21} & \mathbf{U}_{22}\end{bmatrix}. \tag{15}$$

Then, were proved symmetric properties for matrices blocks

$$\mathbf{G}_{22}=\mathbf{G}_{11}^T,\,\mathbf{G}_{12}=-\mathbf{G}_{12}^T,\,\mathbf{G}_{21}=-\mathbf{G}_{21}^T,\ \mathbf{G}_k^{11}=\mathbf{G}_k^{22T},\ \mathbf{G}_k^{12}=-\mathbf{G}_k^{12T},\ \mathbf{G}_k^{21}=-\mathbf{G}_k^{21T}\,;$$

$$\mathbf{S}_{22}=\mathbf{S}_{11}^T,\ \mathbf{S}_{12}=-\mathbf{S}_{12}^T,\ \mathbf{S}_{21}=-\mathbf{S}_{21}^T$$

and were found representations for matrix (9) blocks

$$\left.\begin{array}{l}\mathbf{U}_{11}=0.5\mathbf{I}_n+\mathbf{AS}_{11}-\mathbf{QS}_{21},\ \mathbf{U}_{12}=-\mathbf{AS}_{12}-\mathbf{QS}_{11}^T;\\ \mathbf{U}_{21}=-\mathbf{PS}_{11}-\mathbf{A}^T\mathbf{S}_{21},\mathbf{U}_{22}=\mathbf{I}_n-\mathbf{U}_{11}^T.\end{array}\right\} \tag{16}$$

2.2 Applying resolvent method formulas for urban transportation network

Equations (8), (9)–(16) define numerical procedure of the updated resolvent method. In the paper we consider advantages that give us usage of these equations with matrices (2). Let the following is denoted $\mathbf{R}_d=r^{-1}\mathbf{I}_n,\ \bar{\mathbf{Q}}_d=\mathbf{F}_d\mathbf{Q}_d\mathbf{F}_d^T$.

Then matrices (7) became as

$$\mathbf{\Delta}=2\mathbf{I}_n+\frac{r}{2}\bar{\mathbf{Q}}_d,\ \mathbf{A}=\mathbf{I}_n-2(\mathbf{\Delta}^{-1})^T=\mathbf{I}_n-\left(\mathbf{I}_n+\frac{r}{4}\bar{\mathbf{Q}}_d\right)^{-1},\ \mathbf{Q}=r\mathbf{\Delta}^{-1},\ \mathbf{P}=\mathbf{\Delta}^{-1}\bar{\mathbf{Q}}_d .$$

In the paper the following symmetric properties are proved $\mathbf{A}=\mathbf{A}^T$, $\mathbf{AQ}=\left(\mathbf{AQ}\right)^T$, $\mathbf{PA}=\left(\mathbf{PA}\right)^T$. It yields

$$\mathbf{G}_{11}=\mathbf{QP},\ \mathbf{G}_{12}=\mathbf{G}_{21}=\mathbf{G}_k^{12}=\mathbf{G}_k^{21}=\mathbf{S}_{12}=\mathbf{S}_{21}=\mathbf{0}_{n,n}.$$

The expression for polynomial (12) is obtained as $\delta(x)=\det(x\mathbf{I}_n-\mathbf{G}_{11})$. Then, to find matrix $\mathbf{S}_{11}$ it has sense to apply first linear transformation to $\mathbf{G}=\mathbf{T}\tilde{\mathbf{G}}\mathbf{T}^{-1}$, where matrix $\tilde{\mathbf{G}}$ is a normal Frobenius form matrix. Then, according to (10)

$$\mathbf{S}_{11}=\mathbf{T}\left(\sum_{k=0}^{n-1}\eta_{n-1-k}\tilde{\mathbf{G}}^k\right)\mathbf{T}^{-1}.$$

Finally, matrix (9) can be represented as

$$\mathbf{U}=\begin{bmatrix}0.5\mathbf{I}_n+\mathbf{AS}_{11} & -\mathbf{QS}_{11}^T\\ -\mathbf{PS}_{11} & 0.5\mathbf{I}_n-\mathbf{A}^T\mathbf{S}_{11}^T\end{bmatrix}. \tag{17}$$

and the solution of the equation (8) can be found as a least squares solution

$$\mathbf{U}_2^T\mathbf{U}_2\mathbf{X}+\mathbf{U}_2^T\mathbf{U}_1=\mathbf{0}_{n,n}. \tag{18}$$

Also alternatively in the paper is shown that solution of equation (18), (17) can be found as

$$\mathbf{X}=2\left(\mathbf{AQ}^{-1}\mathbf{A}+\mathbf{P}\right)\mathbf{S}_{11}+\mathbf{Q}^{-1}\mathbf{A}.$$

In a final part of the paper a numerical performance of the proposed approach was evaluated. The evaluation approximately equal to $25n^3$ flops which prove advantages of the proposed approach toward other approaches.

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