SOLUTION BOUNDS

FOR ALGEBRAIC EQUATIONS

IN CONTROL THEORY

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Wisdom sets bounds even to knowledge

Friedrich Nietzsche

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PREFACE

This book is intended for a wide readership including engineers, applied mathematicians, graduate students seeking a comprehensive view of the main results on the estimation of the solutions of four algebraic equations, namely, the continuous- and the discrete-time Lyapunov and Riccati equations. The Lyapunov and Riccati equations arise in many different perspectives, such as: control and system theory, identification, root clustering, differential equations, boundary value problems, power systems control, signal processing, communications, electrical circuits, robust stability, singular systems, decentralized control, linear optimal control and filtering problems, linear dynamic games with quadratic performance index, economic modeling, etc. Due to broad applications, both equations and their numerous modifications remain subjects of active research work.

Many numerical algorithms have been developed for solving the Lyapunov equation (Bartels-Stewart, Hammarling, Hessenberg-Schur methods, etc.) and the Riccati equation (Newton's method, the sign-function method, methods based on eigenvalue computations and their variants, iterative refinement technique, etc.).

Contemporary technology gives rise to some major questions. Much research work has been devoted to the construction of numerically robust algorithms. These methods are characterized by significant memory and computational complexity regardless of the structure of the state matrix. Therefore, the majority of numerical algorithms in linear control theory is restricted to systems of a moderate order. However, a significant number of applications lead to large-scale dynamical models. By no doubt, the direct solution of the Lyapunov, and, especially, of the Riccati equation may turn out to be impractical, since the computational burden increases with the system's dimension. Also, it can be time-consuming, computationally difficult and inaccurate. In other applications, such as stability analysis, it is even not necessary to obtain the exact solution, since some estimates for it are sufficient. The solution bounds can be used to solve various control problems, such as stability analysis of linear perturbed systems and/or systems with pure time delays, robust root clustering, determination of the estimation size error for multiplicative systems, etc. Solution estimates have been used to study robust stability and performance analysis for uncertain stochastic systems, and the estimation of a solution bound based Schur stability margin for real polynomials. The direct computation of the exact solution for the Lyapunov and the Riccati equations may be either impossible, impractical, or, not necessarily required, in some cases. In other cases, estimates can be used as approximations or initial guesses in numerical algorithms.

The book is organized as follows. A summary of proposed since the 1970s various scalar and matrix, lower and upper solution bounds for the considered algebraic equations is presented in Chapter I. The bounds are analyzed with respect to the specific requirements for their validity. The main conclusion is that the available solution estimates are dependable on some rather conservative conditions, e.g., negative definiteness of the symmetric part of the coefficient matrix A for the continuous-time Lyapunov equation, or, maximal singular value of A less than one for the discrete-time Lyapunov equation. Due to this, upper solution bounds for these equations are inapplicable for a large set of stable matrices for which unique positive (semi)-definite solutions for the above mentioned equations exist. Although due to different reasons, the same refers to the Riccati equations.

The advancement made so far in the estimation problem is a topic of this Chapter. Different approaches are presented, discussed and compared, when possible, in order to demonstrate the efficiency and the shortcomings of a particular method. Up to my best knowledge such a detailed summary of all important bounds proposed during the 40 years old history of this research problem is made for the first time.

Motivated by the conservatism in solution estimation, the author suggests a new, alternative way to extend the sets of coefficient matrices for which various lower and upper bounds are valid under less restrictive conditions. The main contributions can be briefly formulated as follows.

1. It has been proved that such extensions can be achieved by taking into account the singular value decomposition of the coefficient matrix for both the continuous-time (Chapter 2) and the discrete-time (Chapter 3) equations. Following this approach various bounds for the extremal eigenvalues, the trace and the solution have been proposed. It is important to say that the respective bounds are valid under relaxed validity constraints. In other words, the available validity sets are extended and generalized by newly defined sets for the continuous- and the discrete-time Lyapunov equation.

2. The author shows how the singular value decomposition approach can be applied to estimate the solutions of the continuous- and the discrete-time Riccati equations. The elimination of some widely used, but not realistic assumptions regarding the state weighting matrix and the control matrix is important.

3. Much attention is paid to the improvement of solution bounds. It is shown how available lower and upper, scalar and matrix bounds can be used to derive new tighter estimates.

4. Unconditional upper bounds for the solutions of all equations are proposed for the first time. Their validity is guaranteed whenever a positive (semi)-definite solution for the respective equation exists.

The bounds proposed in this book are illustrated by eleven numerical examples, including four real data cases in Chapter 4. Namely, these are state space models of an industrial reactor, gas absorber, distillation column and fighter aircraft. The results are analyzed and compared with previously suggested bounds with respect to tightness and validity measured by several error indicators. The obtained results clearly show the superiority of the newly suggested estimates in these cases. Also, the effectiveness of the procedures of iterative improvement is illustrated. The computed bounds coincide with the exact solution in some cases.

The book provides quick and easy references for the solution of different related with solution estimation engineering and mathematical problems. Because both the mathematical development and the applications are considered, it can be useful for solving problems and for research purposes, as well.

Academician Ivan Popchev

NOTATION

$\boldsymbol{R}^p$	set of real *p* x 1 vectors
$\boldsymbol{R}_{m,r}$	set of real *m* x *r* matrices; $\boldsymbol{R}_{m,m} \equiv \boldsymbol{R}_m$
$M > N$	the symmetric matrix *M* – *N* is positive definite
$M \geq N$	the symmetric matrix *M* – *N* is positive semi-definite
$M = [m_{ij}]$	matrix *M* with entries m_{ij}
M^T	transpose of matrix *M*
M^*	conjugate transpose of matrix *M*
M^{-1}	inverse of the nonsingular matrix *M*
$M^{1/2}$	square root of a positive (semi)- definite matrix *M* ; $M = (M^{1/2})^2$
I	identity matrix when the dimension is implicit in the context
I_r	*r* x *r* identity matrix
0	zero matrix when the dimension is implicit in the context
$0_{m,r}$	*m* x *r* zero matrix block; $0_{m,m} \equiv 0_m$
M_S	symmetric part of matrix *M*; $M_S = \frac{1}{2}(M^T + M)$
$diag\{m_{ii}\}$	diagonal matrix with entries m_{ii}
tr(M)	trace of matrix *M*
det(*M*)	determinant of matrix *M*
*rank*M	rank of matrix *M*
ª$_i(M)$	*i*-th eigenvalue of matrix *M*;
$\pm_i(M)$	*i*-th singular value of matrix *M*

The eigenvalues, if real, of an *m* x *m* matrix *M* and the singular values of a *p* x *r* matrix *N* are assumed to be ordered as follows:

$$ª_1(M) \geq ª_2(M) \geq \ldots \geq ª_m(M), \quad \pm_1(N) \geq \pm_2(N) \geq \ldots \geq \pm_s(N), s = \min(p, r)$$

CHAPTER ONE

SOLUTION ESTIMATES: REASONS, HISTORY, PROBLEMS

1.1 INTRODUCTORY REMARKS

The Lyapunov and Riccati equations have been widely used in various fields of modern control and systems theory due to both practical and theoretical reasons.

The Lyapunov equation is named after the Russian mathematician, mechanician and physicist Alexander Lyapunov (1857-1918), who in 1892, in his doctoral dissertation introduced the famous stability theory of linear and nonlinear systems. According to his definition of stability, so called, stability in sense of Lyapunov, one can check the stability of a system by finding out some scalar function (Lyapunov function) and studying the sign definiteness of its total time derivative along the system's motion. Unfortunately, there is no general procedure for finding a Lyapunov function for nonlinear systems, but for linear time-invariant systems, the procedure comes down to the problem of solving a special type of equation, called the Lyapunov equation.

The Lyapunov equation arises in many different perspectives such as:

- control and system theory,
- system identification,
- root clustering,
- linear algebra, optimization, differential, partial differential equations,
- boundary value problems,
- mechanical engineering, power systems control, signal processing,
- large space flexible structures, communications, electrical circuits, etc.,

and therefore, its solution is of great interest [34].

Since linear systems are mathematically very convenient and give fairly good approximations for nonlinear ones, mathematicians and especially engineers often base their analysis on linearized models. Therefore, the solution of the Lyapunov equation gives insight into the behavior of dynamical systems.

The Lyapunov equation is encountered not only in studying the stability, but also in other fields. The quadratic performance index of a linear system is given in terms of the solution of the Lyapunov equation. For stochastic linear systems driven by white noise, the solution of the Lyapunov equation represents the variance of the state vector. Many other control and system theory problems are based on the Lyapunov and/or Lyapunov-like equations, such as: concepts of controllability, observability gramians and estimation design [19], balancing transformation [97], stability robustness to parameter variations [109], [141], reduced order modeling and control [15], [51], [116], power systems [43], filtering with singular measurement noise [43], [44], large space flexible structures [8], etc. The Lyapunov and Lyapunov-like equations also appear in differential games [113], singular systems [90], signal processing [2], [5], [92], differential equations [29], boundary value problems in partial differential equations [69], interpolation problems for rational matrix functions [89]. Another situation when the Lyapunov equation arises is in the design of decentralized control systems. Current research in large scale systems and decentralized control is being directed toward physical systems that, although of large dimension, have sparse system matrices with particular structural forms. Examples of such research are the decentralized control of a freeway traffic corridor, large scale interconnected power systems and the various applications of the concepts of decentralized overlapping control, connective stability and vector Lyapunov functions [129].

Due to broad applications, the Lyapunov equation has been a subject of very active research for the past 60 years. Although the Lyapunov theory has been introduced at the end of the 19-th century, it was not recognized for its vast applications until the 1960s. Since then, it became a major part in control, and systems theory, and other various scientific fields. Around 1965, some researchers like MacFarlane, Barnett and Storey, Chen and Shieh, and Lancaster presented solutions to the Lyapunov equation. In 1970s when growing use of digital computers became part of almost every scientific field, the need for efficient numerical methods was felt. This resulted in celebrated algorithms for numerical solution of the continuous-time algebraic Lyapunov equation (CALE). The classical numerical solution methods for the Lyapunov are the Bartels-Stewart method [10], the Hammarling method [45] and the Hessenberg-Schur method [42]. Extensions of these methods for the generalized Lyapunov equation are presented in [21], [35], [36], [110]. They are based on the preliminary reduction of the matrix (pencil) to the Schur form [41] or the Hessenberg-Schur form [42], calculation of the solution of a reduced form and back transformation. An alternative approach to solve the Lyapunov equation is the sign-function method [12], [75]. Comparison of it to the Bartels-Stewart and Hammarling methods with respect to accuracy and computational cost can be found in [12].

Digital technology in industry also spelled out the need for the solution of the Lyapunov equation for discrete-time systems, called the discrete-time algebraic Lyapunov equation (DALE), which slightly differs from the CALE.

Probably one of the most important results in modern systems theory, both in terms of potential practical and theoretical applications, is the solution of the infinite time least squares problem for stationary linear dynamical systems. It indeed gives a systematic procedure for computing constant feedback control gains for multiple-input systems based on a performance index which admits a simple interpretation in terms of the control effort and the error. There are two main areas in control theory where infinite time least squares minimization problems have been developed. On the one hand there is the standard regulator problem of optimal control theory, and on the other hand, there are the Lyapunov functions which lead, via the so-called Kalman-Yakubovich-Popov Lemma, to the circle criterion and the Popov stability criterion for feedback systems. Although the least squares minimization problem with linear differential constraints has roots going to the very beginnings of calculus of variations, its revival and introduction in control theory may be safely credited to Kalman [53]. In this sense, one should also mention [105], where the least squares technique as a systematic basis for the design of stationary feedback control systems was put forward, the papers by Kalman [54],[55], Popov [114], and the works of Anderson [3] and Anderson and Moore [4], as well.

It is well known that the Riccati equation plays a crucial role in the solution of the optimal control problem under consideration. Willems writes in [137]: “One often gets the impression that this equation in fact constitutes the bottleneck of linear systems theory”. It is named after the nobleman count Jacopo Riccati (1676-1754), born in the Republic of Venice.

The Riccati matrix equation appears, as a consequence of variational problems to be solved, in many fields of applied mathematics, engineering and economic sciences. The Riccati equations belong to the simplest but the most important class of nonlinear equations. Due to its widespread relevance in control and dynamic optimization, the matrix Riccati equation has drawn attention within the mathematical and control theoretic literature. Research in this large field has been considerably stimulated by classical subjects such as [1]:

- linear optimal control and filtering problems with quadratic performance index,
- linear dynamic game and network theory with quadratic performance index,
- stochastic realization theory for linear systems,
- decoupling of linear systems of differential and difference equations,
- spectral factorization of operators,
- singular perturbation theory,
- boundary value problems for systems of ordinary differential equations,
- economic modeling for linear-quadratic control problems, etc.

Although the connections with robustness analysis and dissipation theory for dynamical systems reach back to the 1960s and 1970s, the more recently emerging areas, such as H_∞ - control, have stirred renewed interest in general Riccati equations and their relation with linear matrix inequalities (LMI). The Riccati equation has been extensively studied in [17], [70], [71], [95], [115], [138].

Many numerical algorithms have been developed for solving the Riccati equations and this is still a subject of active research, where also new aspects are coming into account like the treatment of large dimensional or singular systems and the algorithms for parallel computers. Detailed surveys containing extensive lists of references on numerical methods for the Riccati equations are given in [18] and [96]. The Newton's method, the sign-function method, methods based on eigenvalue computations and their variants which exploit the Hamiltonian and sympletic structure of the related eigenvalue problem as well as an iterative refinement technique are discussed in [40].

The Riccati equation can be algebraic, as well as differential or difference equation. As in the case with the Lyapunov equation, the algebraic Riccati equation can appear in continuous-time, or in discrete-time form, usually abbreviated CARE and DARE, respectively.

In numerical problems it is very important to study the sensitivity of the solution to perturbations in the output data and to bound errors in the computed solution. There are several papers concerned with perturbation theory and the backward error bounds for CALE, e.g. see [46], [47], [132] and the references therein. Sensitivity analysis for the generalized Lyapunov and Riccati equations has been presented in [48], [67].

1.2 WHY ESTIMATES?

Complex contemporary technology and its various applications lead to some important and not easy to answer questions. In the last 2-3 decades, much research has addressed the construction of numerically robust algorithms that arise in context with linear systems. Such problems are, e.g., optimal control, robust control, system identification, game theory, model reduction and filtering. However, these methods generally have at least a memory complexity $O(n^2)$ and computational complexity $O(n^3)$ regardless whether or not the n x n system matrix A is sparse or otherwise structured. Therefore, the majority of numerical algorithms in linear control theory is restricted to systems of a moderate order. Of course, the upper limit for this order depends on the problem to be solved as well as on the particular computing environment and may vary between few hundred and few thousands [14]. The most popular approaches with cubic complexity are surveyed in [13], [24], [96], [112], [130]. An extensive study of the computational complexity results in systems and control theory can be found in [16]. It provides a tutorial introduction to some key concepts from the theory, highlighting their relevance to systems and control theory and surveys the recent research activity in these fields.

However, a significant number of applications lead to dynamical systems of larger order. Large-scale systems may arise from the semidiscretization of (possibly) linearized partial differential equations by means of finite differences or finite elements [9], [52], [76].

Another sources for such systems are circuit design and simulation [32], [33], or large space mechanical structures [38], [107] and the application of the Kalman filter to the problem of assimilating atmospheric data [22]. With some simplifying assumptions the error covariance of the estimate of the state of the atmosphere satisfies the DALE, which has two distinguishing properties. The system comes from the discretization of a three dimensional continuum problem, the dimension of the matrices is large, typically of order $n = 10^6$ and the direct treatment of the DALE is impossible [133]. In principle one can obtain the solution of the CALE by using the skew-symmetric matrix approach and solving a system of $0.5n(n-1)$ linear algebraic equations [128], or by transforming the system matrix into some canonical forms like the Jordan form [93], or companion form[131]. However such approaches require large memory and computer processing time of the order of « n^6 , where « is the time for one multiplication or division. This is a very large number for large n and becomes impractical even for moderate $n > 10$. The important algorithms of Bartels and Stewart, Golub and Hammarling require $O(n^2)$ memory locations and $O(n^3)$ multiplications. *By* no doubt, the direct solution of the Lyapunov and, especially, of the Riccati equations may turn out to be impractical even in cases when the respective solution can be found numerically, since the computational burden increases with the systems dimension [30], it can be time-consuming and computationally difficult [25]. One of the numerical algorithms for the solution of the CARE is a Schur-type method, which is described in detail in [7], [78], [108]. This method is very reliable and is used in $MATRIX_X$, $CTRL-C$ and *MATLAB*. But the number of operations required for the solution is estimated by more than $75n^3$ [59], [60], which takes considerable time when n is high even though powerful computational environment may be used.

In other applications, such as stability analysis, it is even not necessary to know the exact solution because an estimate for it is sufficient. Also, if the parameters of the system are uncertain it is not possible to obtain the exact solution for robust stability analysis and knowing some bounds on the solution can be useful. Furthermore, the solution bounds can be applied to solve many control problems such as stability analysis of linear perturbed systems and/or systems with pure delays [86], [136], robust root clustering [85], [142], determination of the size of the estimation error for multiplicative systems [68] and so on.

Solution estimates have been used to study robust stability and performance analysis for uncertain stochastic systems in [31] and a solution bound based Schur stability margin for real polynomials has been estimated in [100]. Therefore, computing the exact solution of the Lyapunov and/or of the Riccati equations may be either impossible, or impractical, or even not necessarily required, in some practical cases.

Finally, proper estimates can be very useful even when exact solutions are computable and necessarily required, since they may be used as approximations of the solution or initial guesses in the numerical algorithms for the exact solution [74].

The significance of these equations as important and powerful tools used in various fields of systems and control theory and the role they play in the solution of many practical problems is clear. The above mentioned difficulties arising in some specific applications, explain completely why the estimation problem for the algebraic Lyapunov and Riccati equations has attracted such a considerable attention in the past five decades and still remains a field of active research.

1.3 LYAPUNOV AND RICCATI ALGEBRAIC EQUATIONS

1.3.1 THE CONTINUOUS-TIME CASE

According to Lyapunov's theory, the stability of dynamical systems can be determined in terms of the so called Lyapunov functions. This can be done for systems both in continuous- and discrete-time domains. Consider a continuous-time system:

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{1.1}$$

where $x(t) \in \boldsymbol{R}^{\mathrm{n}}$ is the state vector and $A \in \boldsymbol{R}_n$ is the state matrix. The following stability definitions are well known.

Definition 1.1 The system (1.1) is asymptotically stable if all eigenvalues of the state matrix lie in the open left half of the complex plane.

The celebrated Lyapunov stability theorem is formulated as follows (e.g. see [57]).

Theorem 1.1 An equilibrium point of a time-invariant system is asymptotically stable if there exists a continuously differentiable scalar function *v*(*x*) such that along the system trajectories the following set of inequalities is satisfied:

$$v(x) > 0, \quad \forall x \neq 0$$

$$\dot{v}(x) = \frac{\partial v}{\partial x}\frac{dx}{dt} = -w(x) < 0, \quad \forall x \neq 0$$

It is easy to show that for a linear system (1.1), a Lyapunov function can be chosen as a quadratic one, i.e., $v(x) = x^T(t)Px(t), \quad P > 0$, which with use of (1.1) leads to

$$\dot{v}(x) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) = x^T(t)(A^TP + PA)x(t)$$

Lyapunov showed that the function $w(x)$ can be *a priori* chosen as quadratic form, i.e., $w(x) = x^T(t)Qx(t), Q > 0$, which results in

$$\dot{v}(x) = x^T(t)(A^TP + PA)x(t) = -x^T(t)Qx(t), \forall x(t) \neq 0 \Leftrightarrow A^TP + PA + Q = 0$$

Now, the original stability problem is reduced to solving the so called CALE

$$A^TP + PA = -Q, \quad Q > 0 \qquad (1.2)$$

The condition for existence of a unique solution states that if $a_i(A), i = 1,..,n$, denotes the *i*-th eigenvalue of *A*, then $a_i(A) + a_j(A) \neq 0, i, j = 1,..,n$, i.e., no two eigenvalues of *A* add up to zero [50]. This condition is obviously satisfied if *A* is asymptotically stable in the continuous-time domain.

The Lyapunov stability theory for linear continuous-time invariant systems is formulated as follows [19].

Theorem 1.2 The system (1.1) is asymptotically stable if and only if for any positive definite matrix *Q* a unique positive definite solution *P* of the CALE (1.2) exists. In addition, if $Q = C^TC \geq 0$, then (1.1) is asymptotically stable system if and only if (*C*, *A*) is an observable pair and the CALE has a unique positive definite solution.

If the solution *P* is a positive definite matrix, the function $v(x) = x^T(t)Px(t)$, is said to be a Lyapunov function for the system (1.1).

The unique positive (semi)-definite solution of the CALE is given by

$$P = \int_0^\infty e^{A^Tt}Qe^{At}dt$$

and the system performance is usually measured with respect to the criterion

$$J(x_0) = \int_0^\infty x^T(t)Qx(t)dt, \quad Q \geq 0$$

Since the solution of the vector differential equality (1.1) is $x(t) = e^{At}x_0$, then

$$J(x_0) = \int_0^\infty x_0^T e^{A^T t} Q e^{At} x_0 dt = x_0^T P x_0 \tag{1.3}$$

This admits to evaluate easily the system performance for arbitrarily given positive (semi)-definite right-hand side matrix in (1.2) and initial state vector.

Consider a continuous-time invariant control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{1.4}$$

$$y(t) = Cx(t)$$

where $x(t) \in \boldsymbol{R}^n$, $u(t) \in \boldsymbol{R}^m$ and $y(t) \in \boldsymbol{R}^r$ are the state, control and output vectors, respectively, and $A \in \boldsymbol{R}_n$, $B \in \boldsymbol{R}_{n,m}$, $C \in \boldsymbol{R}_{r,n}$ are given matrices. Consider the quadratic performance index

$$J(u, x_0) = \int_0^\infty [(y^T(t)y(t) + u^T(t)Ru(t)]dt \to min \tag{1.5}$$

where R is a positive definite control weighting matrix. If (A, B) and (C, A) are stabilizable and detectable pairs, respectively, then (C, A, B) is said to be a regular triple. It is desired to determine a control law $u(t) = - Kx(t)$, which minimizes (1.5) and stabilizes the close-loop state matrix $A_c = A - BK$.

Theorem 1.3 [67] Let (C, A, B) be a regular triple. The control law that minimizes the performance index (1.5) for every initial state vector is realized as a state feedback

$u(t) = -R^{-1}B^T Px(t)$, where P is the unique positive semi-definite solution of the CARE

$$A^T P + PA - PBR^{-1}B^T P = -Q, \quad Q = C^T C \tag{1.6}$$

In this case, the close loop state matrix $A_c = A - BR^{-1}B^T P$ is asymptotically stable and

$$J(u, x_0) = x_0^T P x_0 \tag{1.7}$$

In addition, if (C, A) is a completely observable pair, then the solution P of the CARE is a strictly positive definite matrix.

At the same time the CARE may have other solutions, which necessarily are not positive (semi)-definite and not stabilizing, including nonsymmetric ones, as well.

1.3.2. THE DISCRETE-TIME CASE

Consider a discrete-time system:

$$x(k+1) = Ax(k), \quad x(0) = x_0 \tag{1.8}$$

where $x(k) \in \boldsymbol{R}^{n}$ is the state vector and $A \in \boldsymbol{R}_n$ is the state matrix. The following stability definition is well known.

Definition 1.2 The system (1.8) is asymptotically stable if all eigenvalues of the state matrix lie inside the unit circle.

Dual to Theorems 1.1 and 1.2 can be stated for the stability of the linear discrete-time system (1.8). Consider a quadratic Lyapunov function *v*(*k*), which for the system stability must satisfy [106]

$$v(k) = x^T(k)Px(k), \quad P > 0,$$

$$\Delta v(k) = v(k+1) - v(k) = -w(k),\ w(k) > 0$$

where, as in the continuous-time case, the function *w*(*k*), can be *a priori* chosen as a quadratic form, i.e., $w(k) = x^T(k)Qx(k),\ Q > 0$. In this case,

$$v(k+1) - v(k) = x^T(k+1)Px(k+1) - x^T(k)Px(k) = x^T(k)(A^TPA - P)x(k) = -w(k),$$

or,

$$\Delta v(k) = x^T(k)(A^TPA - P)x(k) = -x^T(k)Qx(k),\ \forall x(k) \neq 0 \Leftrightarrow A^TPA - P + Q = 0$$

which obviously converts the stability problem into the solution problem for the DALE

$$A^TPA - P = -Q \tag{1.9}$$

The condition for existence of a unique solution for (1.9) states that if $\lambda_i(A),\ i = 1,..,n$, denotes the *i*-th eigenvalue of *A*, then $\lambda_i(A)\lambda_j(A) \neq 1,\ i, j = 1,..,n$, i.e., no two eigenvalues of *A* have product equal to one [34]. This condition is obviously satisfied if *A* is asymptotically stable in the discrete-time domain.

The Lyapunov stability theory for linear discrete-time systems is formulated as follows.

Theorem 1.4 The system (1.8) is asymptotically stable if and only if for any positive definite matrix *Q* a unique positive definite solution *P* of the DALE (1.9) exists. In addition, if $Q = C^TC \geq 0$, then (1.8) is asymptotically stable system if and only if (*C*, *A*) is an observable pair and the DALE has an unique positive definite solution.

If the solution P is a positive definite matrix, the function $v(k) = x^T(k)Px(k),$ is said to be a Lyapunov function for the system (1.1). The unique positive (semi)-definite solution of the DALE is given by

$$P = \sum_{k=0}^{\infty} (A^k)^T Q A^k \tag{1.10}$$

and the system performance is usually measured with respect to the criterion

$$J(x_0) = \sum_{k=0}^{\infty} x^T(k)Qx(k), \quad Q \geq 0$$

Since the solution of the vector differential equality (1.1) is $x(k) = A^k x_0$, then

$$J(x_0) = \sum_{k=0}^{\infty} x_0^T A^{kT} Q A^k x_0 = x_0^T P x_0 \tag{1.11}$$

This admits to evaluate easily the system performance in the discrete-time case for arbitrary positive (semi)-definite right-hand side matrix in (1.9) and initial state vector.

Consider a discrete-time invariant control system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \tag{1.12}$$

$$y(\mathrm{k}) = Cx(k)$$

where $x(k) \in \boldsymbol{R}^n$, $u(k) \in \boldsymbol{R}^m$ and $y(k) \in \boldsymbol{R}^r$ are the state, control and output vectors, respectively, and $A \in \boldsymbol{R}_n$, $B \in \boldsymbol{R}_{n,m}$, $C \in \boldsymbol{R}_{r,n}$ are given matrices. Consider the quadratic performance index

$$J(u, x_0) = \sum_{k=0}^{\infty} [y^T(k)y(k) + u^T(k)Ru(k)] \to min \tag{1.13}$$

where R is a positive definite control weighting matrix. Let (C, A, B) be a regular triple. It is desired to determine a control law $u(k)$ = - $Kx(k)$, which minimizes (1.13) and stabilizes the close-loop state matrix $A_c = A - BK$.

Theorem 1.5 [67] Let (C, A, B) be a regular triple. The control law that minimizes the performance index (1.13) for every initial state vector is realized as a state feedback $u(k) = -(R + B^T PB)^{-1} B^T PAx(k)$, where P is the unique positive semi-definite solution of the DARE

$$A^T PA - P - A^T PB(R + B^T PB)^{-1} B^T PA = -Q, \quad Q = C^T C \tag{1.14}$$

In this case, the close loop state matrix $A_c = A - B(R + B^T PB)^{-1} B^T PA$ is asymptotically stable and

$$J(u, x_0) = x_0^T P x_0 \tag{1.15}$$

In addition, if (C, A) is a completely observable pair, then the solution *P* of the DARE is a strictly positive definite matrix.

From now on, it is assumed that the considered algebraic equations have unique positive (semi)-definite solutions.

Theorems 1.2 and 1.4 state the necessary and sufficient conditions for stability of the linear time-invariant systems (1.1) and (1.8), respectively, in terms of the CALE and the DALE solution matrices, which allow to determine Lyapunov functions for them, as well. Stability of any of these systems is equivalent to stability of the respective state matrices. A more general formulation of the stability problem is given by the following well known results.

Theorem 1.6. A matrix *A* is stable in the continuous-time sense if and only if there exists a positive definite matrix *P*, such that $A^T P + PA < 0$. If $A^T P + PA \le 0$ and *A* is a stable matrix, then *P* is a positive (semi)-definite matrix.

Theorem 1.7. A matrix *A* is stable in the discrete-time sense if and only if there exists a positive definite matrix *P*, such that $A^T PA - P < 0$. If $A^T PA - P \le 0$ and *A* is a stable matrix, then *P* is a positive (semi)-definite matrix.

Definition 1.3. If matrix *P* satisfies the assumptions of Theorem 1.6, or Theorem 1.7, it is said to be a Lyapunov matrix for *A*.

1.4 SUMMARY OF BOUNDS

Many bounds for the positive (semi)-definite solutions of the algebraic Lyapunov equations (1.2), (1.9) and the Riccati equations (1.6), (1.14) have been reported since the first results were obtained about fifty years ago. All bounds reported during the first two decades were summarized in [74], [101]. Since then, research in this area has brought out many new results.

Depending on their type, the bounds can be matrix or scalar, lower or upper. As a measure of the "size" of the solution, the following scalar estimates are proposed: for the eigenvalues (including extremal eigenvalues), the trace and the determinant. The maximal eigenvalue apparently means the respective solution gain. The *i*-th eigenvalue provides important finer information about the solution. In view of the relations, the determinant and the trace give the geometric and the arithmetic mean eigenvalue. Of all these measurements, the matrix estimates are the most general and the most valuable, since they can be used to obtain all other ones.

During the first two decades (1974-1994) the following important estimates for the extremal eigenvalues, the trace and the solution have been derived. Note, that some of the bounds are valid only under specific assumptions, which are explicitly specified and will be commented later on.

Remark 1.1 It is usually assumed that $R = I$ for the CARE and the DARE. In fact, this is not a restriction at all, since R is a positive definite matrix, $R^{-1/2}$ exists and the equations (1.6) and (1.14) can be equivalently rewritten as follows:

$$A^T P + PA - P\tilde{B}\tilde{B}^T P = -Q, \quad Q = C^T C,$$

$$A^T PA - P - A^T P\tilde{B}(I + \tilde{B}^T P\tilde{B})^{-1}\tilde{B}^T PA = -Q, \quad Q = C^T C$$

where $\tilde{B} = BR^{-1/2}$. Without any loss of generality, in what follows, the control weighting matrix is assumed to be identity.

Solution bounds for CALE (1.2):

$$\lambda_n(P) \geq \frac{\lambda_n(Q)}{2\sigma_1(A)} \quad [127] \tag{1.16}$$

$$\lambda_n(P) \geq \frac{1}{-2\lambda_n(A_s Q^{-1})} \ ; \text{ if } Q > 0 \ \ [139] \tag{1.17}$$

$$\lambda_n(P) \geq \frac{\lambda_n(Q)}{-2\lambda_n(A_s)} \quad [99] \tag{1.18}$$

$$\lambda_1(P) \leq \frac{1}{-2\lambda_1(A_s Q^{-1})} \ ; \text{ if } A_s < 0, Q > 0 \quad [139] \tag{1.19}$$

$$tr(P) \geq \frac{tr(Q)}{-2\lambda_n(A_s)} \quad [135], [102] \tag{1.20}$$

$$tr(P) \geq \frac{tr(Q)}{-2tr(A_s)} \quad [135] \tag{1.21}$$

$$tr(P) \geq \frac{[\sum_{i=1}^{n} \lambda_i^{1/2}(Q)]^2}{-2tr(A_s)} \quad [63] \qquad (1.22)$$

$$tr(P) \leq -\sum_{i=1}^{n} \frac{\lambda_i(Q)}{2\lambda_i(A_s)} \text{; if } A_s < 0 \quad [65] \qquad (1.23)$$

$$tr(P) \leq \frac{tr(Q)}{-2\lambda_1(A_s)} \text{; if } A_s < 0 \quad [102] \qquad (1.24)$$

$$P \geq \frac{1}{-\lambda_n[Q^{-1/2}(A^TQ+QA)Q^{-1/2})}Q \text{; if } Q > 0 \;\; [39] \qquad (1.25)$$

$$P \leq \frac{1}{-\lambda_1[Q^{-1/2}(A^TQ+QA)Q^{-1/2})}Q \text{; if } A^TQ+QA < 0 \;\; [39] \qquad (1.26)$$

Solution bounds for CARE (1.6):

$$\lambda_n(P) \geq \frac{\lambda_n(Q)}{\sigma_1(A)+[\sigma_1^2(A)+\sigma_1^2(B)\lambda_n(Q)]^{1/2}} \quad [73] \qquad (1.27)$$

$$\lambda_n(P) \geq \frac{1}{-\lambda_n(A_sQ^{-1})+[\lambda_n^2(A_sQ^{-1})+\lambda_1(BB^TQ^{-1})]^{1/2}} \text{; if } Q > 0 \quad [73] \qquad (1.28)$$

$$\lambda_n(P) \geq \frac{\lambda_n[Q(BB^T)^{-1}]}{-\lambda_n[A_s(BB^T)^{-1}]+\{\lambda_n^2[A_s(BB^T)^{-1}]+\lambda_1[Q(BB^T)^{-1}]\}^{1/2}} \text{; if } BB^T > 0 \quad [139] \qquad (1.29)$$

$$\lambda_1(P) \leq \frac{1}{-\lambda_1(A_sQ^{-1})+[\lambda_1^2(A_sQ^{-1})+\lambda_n(BB^TQ^{-1})]^{1/2}} \text{; if } Q > 0 \text{ and } BB^T > 0 \quad [139] \qquad (1.30)$$

$$\lambda_1(P) \leq \frac{\lambda_1(Q)}{-\sigma_1(A)+[\sigma_1^2(A)+\sigma_n^2(B)\lambda_1(Q)]^{1/2}} \text{; if } BB^T > 0 \quad [56], [98] \qquad (1.31)$$

$$\lambda_1(P) \leq \frac{\lambda_1(Q)}{-|\lambda_1(A_s)|+[|\lambda_1(A_s)|^2+\lambda_n(BB^TQ^{-1})]^{1/2}} \text{; if } BB^T > 0 \quad [140] \qquad (1.32)$$

$$\lambda_1(P) \le \frac{\lambda_1[Q(BB^T)^{-1}]}{-\lambda_1[A_s(BB^T)^{-1}] + \{\lambda_1^2[A_s(BB^T)^{-1}] + \lambda_n[Q(BB^T)^{-1}\}^{1/2}}\text{; if } BB^T > 0 \qquad [58] \qquad (1.33)$$

$$tr(P) \ge \frac{tr(Q)}{-tr(A) + \{[tr(A)]^2 + \sigma_1^2(B)tr(Q)\}^{1/2}}; \qquad [134] \qquad (1.34)$$

$$tr(P) \ge \frac{tr(Q)}{-\lambda_n(A_s) + [\lambda_n^2(A_s) + \sigma_1^2(B)tr(Q)]^{1/2}}; \qquad [96] \qquad (1.35)$$

$$tr(P) \le \frac{tr(Q)}{-\lambda_1(A_s) + [\lambda_1^2(A_s) + n^{-1}\sigma_n^2(B)tr(Q)]^{1/2}}\text{; if } BB^T > 0 \qquad [135] \qquad (1.36)$$

$$tr(P) \le \frac{tr[Q(BB^T)^{-1}]}{-\lambda_1(A_s) + \{\lambda_1^2(A_s) + 2n^{-1}tr[Q(BB^T)^{-1}\}^{1/2}}\text{; if } BB^T > 0 \qquad [61] \qquad (1.37)$$

Solution bounds for DALE (1.9):

$$\lambda_n(P) \ge \frac{\lambda_n(Q)}{1-\sigma_n^2(A)} \qquad [139] \qquad (1.38)$$

$$\lambda_1(P) \le \frac{\lambda_1(Q)}{1-\sigma_1^2(A)}\text{; if } AA^T < I \qquad [139] \qquad (1.39)$$

$$tr(P) \ge \frac{tr(Q)}{1-\sigma_n^2(A)}; \qquad [103] \qquad (1.40)$$

$$tr(P) \ge \frac{n\lambda_n(Q)}{1-n^{-1}\sum_{i=1}^{n}|\lambda_i(A)|^2}; \qquad [72] \qquad (1.41)$$

$$tr(P) \le \frac{tr(Q)}{1-\sigma_1^2(A)}\text{; if } AA^T < I \qquad [103] \qquad (1.42)$$

Solution bounds for DARE (1.14):

$$\lambda_n(P) \ge \frac{2\lambda_n(Q)}{N + [N^2 + 4\sigma_1^2(B)\lambda_n(Q)]^{1/2}};\ N = 1-\sigma_n^2(A) - \sigma_1^2(B)\lambda_n(Q); \qquad [139] \qquad (1.43)$$

$$\lambda_1(P) \le \frac{2\lambda_1(Q)}{N + [N^2 + 4\sigma_n^2(B)\lambda_1(Q)]^{1/2}};\ N = 1-\sigma_1^2(A) - \sigma_n^2(B)\lambda_1(Q)\text{, if } BB^T > 0 \quad [139](1.44)$$

$$\lambda_1(P) \le \frac{\sigma_1^2(A) + 4\sigma_n^2(B)\lambda_1(Q)}{[4-\sigma_1^2(A)]\sigma_n^2(B)}, \text{ if } AA^T < 4I \qquad [37] \qquad (1.45)$$

$$tr(P) \ge \frac{2n^2\lambda_n(Q)}{-N + [N^2 + 4\sigma_1^2(B)\lambda_n(Q)]^{1/2}};\ N = \sum_{i=1}^{n} \left|\lambda_i(A)\right|^2 + tr(BB^TQ) - n \qquad [72] \qquad (1.46)$$

$$tr(P) \ge \frac{2tr(Q)}{-N + [N^2 + 4\sigma_1^2(B)tr(Q)]^{1/2}};\ N = -1 + \sigma_n^2(A) + \sigma_1^2(B)tr(Q) \qquad [104] \qquad (1.47)$$

$$tr(P) \le \frac{2tr(Q)}{N + [N^2 + 4n^{-1}\sigma_n^2(B)tr(Q)]^{1/2}};\ N = 1 - \sigma_1^2(A) - \sigma_n^2(B)\lambda_1(Q);\text{if } BB^T > 0\ [66] \qquad (1.48)$$

$$P \ge A^T(Q^{-1} + BB^T)^{-1}A + Q\ ;\ \text{if } Q > 0 \qquad [62] \qquad (1.49)$$

$$P \le A^T(BB^T)^{-1}A + Q;\ \text{if } BB^T > 0 \qquad [62] \qquad (1.50)$$

In view of (1.16)-(1.50), the following general conclusions can be drawn.

1. Bounds for CALE. Validity is guaranteed for the minimal eigenvalue lower bound (1.17) and the lower solution matrix bound (1.25) only if *Q* is a positive definite matrix. Although this is not explicitly required for the bounds (1.16) and (1.18), if Q is a positive semi-definite matrix, then the trivial estimate $\lambda_n(P) \ge 0$ is obtained. The upper bounds for the maximal eigenvalue (1.19) and the trace (1.23), (1.24), of the solution *P* are restricted only to the case in which the symmetric part of the coefficient matrix *A* is negative definite. The upper matrix bound (1.26) is valid only if the right-hand side matrix satisfies the inequality $A^TQ + QA < 0$ which obviously means that *Q* is a positive definite matrix, by necessity. It is hard to say which requirement is more conservative. It is well known that stability of a given matrix does not imply negative definiteness of its symmetric part and $v(x) = x^T(t)Qx(t)$ may not be a Lyapunov function for system (1.1), even though it is positive, in the general case. The lower trace estimates (1.20) and (1.21) and (1.22) are valid whenever a positive semi-definite solution *P* exists.

2. Bounds for CARE. The three lower bounds for the solution minimal eigenvalue (1.27), (1.28) and (1.29) provide the trivial estimate $\lambda_n(P) \ge 0$, if *Q* is not a positive definite matrix. In addition, (1.29) is valid only if $BB^T > 0$. All upper bounds for the solution maximal eigenvalue (1.30), (1.31), (1.32) and (1.33) are valid only if $BB^T > 0$. The same

refers to the upper trace estimates (1.36) and (1.37). The bound (1.30) requires that $Q > 0$, as well.

3. Bounds for DALE. If Q is not a strictly positive definite matrix, then bound (1.38) provides the estimate $\lambda_n(P) \geq 0$. The upper bounds for the solution maximal eigenvalue (1.39) and the solution trace (1.42) are valid only if $AA^T < I \Leftrightarrow \sigma_1(A) < 1$. This requirement corresponds to the restriction $A_s < 0$ in the continuous time case. It is well known that stability of a given matrix in discrete-time sense does not necessarily imply that its largest singular value is less than one. Both lower trace bounds (1.40) and (1.41) are valid, whenever a positive semi-definite solution of the DALE exists.

4. Bounds for CARE. The lower bound (1.43) for the solution minimal eigenvalue is trivial, i.e., $\lambda_n(P) \geq 0$, if Q is not a strictly positive definite matrix. The application of the upper estimates for the maximal eigenvalue of the solution matrix presupposes that $BB^T > 0$, for bound (1.42), and $AA^T < 4I \Leftrightarrow \sigma_1(A) < 2$, for bound (1.45). The lower and upper matrix bounds (1.49) and (1.50) are valid if $Q > 0$ and $BB^T > 0$, respectively. Both lower trace bounds (1.46) and (1.47) are applicable whenever a positive (semi)-definite solution exists.

This brief analysis of the derived during the first period in the history of solution estimation bounds shows clearly several important facts.

(i) All lower bounds for the minimal solution eigenvalue provide trivial estimates if Q is not a strictly positive definite matrix.

(ii)The upper bounds for the maximal eigenvalue and the trace are derived under some rather conservative assumptions: $A_s < 0$ (CALE), $AA^T < I$ (DALE), $BB^T > 0$ (CARE and DARE), etc.

It is well known that in real life applications the number of states is always greater than the number of control inputs, i.e., $n > m$, sometimes even $n >> m$. Therefore, BB^T is singular even if B is a full rank matrix. The exact conditions for the existence of unique positive definite solutions for the four algebraic equations are clearly stated by Theorems 1.2, 1.3, 1.4 and 1.5. Therefore, there exist too many cases in which it is not possible to get valid bounds for the respective positive definite solution matrices and their extreme eigenvalues and traces. By no doubt, conservativeness with respect to various validity restrictions imposed on the parameters of the equations constitutes the main shortcoming of the estimation problem.

After 1995, during the second period of solution estimation, more attention was naturally paid to the problem of conservatism reduction.

An attempt to overcome the main difficulty concerning upper eigenvalue and trace bounds for the solution of the CALE was firstly made in [30] in 1997. From Theorem 1.7 it follows that for any stable matrix A there exists some positive definite matrix T, such that

$$A^T T^{-2} + T^{-2}A < 0 \Leftrightarrow TA^T T^{-1} + T^{-1}AT = \tilde{A}^T + \tilde{A} < 0 \Leftrightarrow \lambda_1(\tilde{A}_s) < 0, \quad \tilde{A} = T^{-1}AT \tag{1.51}$$

Pre- and post-multiplication of equation (1.2) by matrix T results in the modified CALE

$$\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} = -\tilde{Q}, \quad \tilde{P} = TPT, \quad \tilde{Q} = TQT, \quad \lambda_1(\tilde{A}_s) < 0 \tag{1.52}$$

for which upper bounds for the maximal eigenvalue and the trace of the solution $\tilde{P}$ can be obtained. Then, having in mind that

$$\lambda_1(\tilde{P}) = \lambda_1(PT^2) \geq \lambda_1(P)\lambda_n(T^2) = \lambda_1(P)\lambda_n^2(T)$$

and

$$tr(\tilde{P}) = tr(PT^2) \geq tr(P)\lambda_n(T^2) = tr(P)\lambda_n^2(T)$$

the following new upper bounds have been obtained in [30]:

$$\lambda_1(P) \leq \frac{\lambda_1(QT^2)}{-2\lambda_n^2(T)\lambda_1(\tilde{A}_s)}; \text{ if } \lambda_1(\tilde{A}_s) < 0, \quad \tilde{A} = T^{-1}AT \tag{1.53}$$

$$tr(P) \leq \frac{tr(QT^2)}{-2\lambda_n^2(T)\lambda_1(\tilde{A}_s)}; \text{ if } \lambda_1(\tilde{A}_s) < 0, \quad \tilde{A} = T^{-1}AT \tag{1.54}$$

$$tr(P) \leq \frac{1}{-2\lambda_n^2(T)} \sum_{i=1}^{n} \frac{\lambda_i(QT^2)}{\lambda_i(\tilde{A}_s)}; \text{ if } \lambda_1(\tilde{A}_s) < 0, \quad \tilde{A} = T^{-1}AT \tag{1.55}$$

Remark 1.2. Bounds (1.53)-(1.55) generalize some previous estimates, since if $\lambda_1(A_s) < 0$**,** then $T = I$ is a Lyapunov matrix for A. Theoretically speaking, a matrix T satisfying the validity constraints always exists and in this sense the above three upper bounds are always valid, but a new computational problem arises. If T is obtained via some external procedure, e.g., solving a LMI, it may turn out that the computational complexity required in this case is comparable with the one needed for the direct solution, which may undermine this approach.

Definition 1.4. If a given solution bound is expressed entirely in terms of the parameters of the respective equation, i.e., the matrices *A*, *Q*, *B*, *R* and their eigenvalues, singular values, traces, etc., it is said to be an internal bound. Otherwise, it is called external.

E.g., all bounds (1.16) – (1.50) are internal, while, for $T \neq I$, the bounds (1.53)-(1.55) are external upper solution bounds.

Consider the matrix identity

$$(X^{-1} + YZ)^{-1} = X - XY(I + ZXY)^{-1} ZX \tag{1.56}$$

where X, Y and Z are some matrices with appropriate dimensions such that the above inverse matrices exist. By application of (1.56) for $X = P$, $Y = B$ and $Z = B^T$ one gets

$$(P^{-1} + BB^T)^{-1} = P - PB(I + B^T PB)^{-1} B^T P$$

This allows rewriting the DARE (1.14) as

$$P = A^T (P^{-1} + BB^T)^{-1} A + Q \tag{1.57}$$

It was proved in [65] that the matrix inequalities $0 < P$, $0 \leq P_L \leq P \leq P_U$ imply

$$(I + P_L BB^T)^{-1} P_L \leq (P^{-1} + BB^T)^{-1} = (I + PBB^T)^{-1} P \leq (I + P_U BB^T)^{-1} P_U \tag{1.58}$$

If A is a stable in the discrete-time sense matrix, then the DALE (1.9) has a positive (semi)-definite solution P_U for any given positive (semi)-definite matrix *Q*. It is easy to see, that the difference matrix $P_\Delta = P_U - P$ is the solution of the DALE-type equation

$$P_\Delta = A^T P_\Lambda A + A^T PB(I + BPB^T)^{-1} B^T PA \geq A^T P_\Lambda A \Rightarrow 0 \geq A^T P_\Delta A - P_\Delta \Leftrightarrow P_\Delta \geq 0$$

in accordance with Theorem 1.7. From (1.57) it follows that matrix *Q* is a lower matrix bound for the DARE (1.14) solution, i.e. $P \geq Q$. These facts and the inequalities (1.58) are essential for the derivation of the following lower and upper matrix solution bounds for the DARE [59]

$$A^T (I + QBB^T)^{-1} QA + Q \leq P \leq A^T (I + P_U BB^T)^{-1} P_U A + Q \tag{1.59}$$

The lower bound is remarkable since it is valid whenever (1.14) has a positive definite solution. Note, that if *Q* is a positive definite matrix it yields the lower bound (1.49). As far as the upper bound in (1.59) is concerned, it is applicable for arbitrary matrices *B* and *Q*, but besides the fact that P_U is a solution of the DALE (1.9), its validity requirement (*A* must be a stable matrix) is a rather conservative one.

Motivated by the fact that upper solution bounds for the Riccati-type equations require usually the condition $BB^T > 0$, another attempt to overcome this not realistic assumption was made for the derivation of upper matrix bounds for the CARE (1.6) in [60]. Since the pair (A, B) is stabilizable, there exists some matrix $K \in \boldsymbol{R}_{m,n}$ such that the close-loop matrix $A_C = A - BK$ is stable. It was proved that the solution of the CARE satisfies $P \le P_U$, where the upper matrix bound is the unique positive (semi)-definite solution of the CALE

$$A_C^T P_U + P_U A_C = -\tilde{Q}, \quad \tilde{Q} = Q + K^T K \tag{1.60}$$

The authors observed that the solution of the CALE is obtained much more easily than the solution of the CARE, but the problem of choosing K which provides the tightest in some sense bound is open. They also proposed upper bounds for the maximal eigenvalue and the trace of the solution P of the CARE, expressed in terms of the respective bounds for the solution of the CALE (1.60) given by (1.53) and (1.55), i.e.:

$$\lambda_1(P) \le \lambda_1(P_U) \le \frac{\lambda_1(\tilde{Q}T^2)}{-2\lambda_n^2(T)\lambda_1(\tilde{A}_{Cs})}; \text{ if } \lambda_1(\tilde{A}_{Cs}) < 0, \quad \tilde{A}_C = T^{-1}A_C T \tag{1.61}$$

$$tr(P) \le tr(P_U) \le \frac{1}{-2\lambda_n^2(T)} \sum_{i=1}^{n} \frac{\lambda_i(\tilde{Q}T^2)}{\lambda_i(\tilde{A}_{Cs})}; \text{ if } \lambda_1(\tilde{A}_{Cs}) < 0, \quad \tilde{A}_C = T^{-1}A_C T \tag{1.62}$$

If A is a stable matrix, then $K = 0$ and $\tilde{Q} \equiv Q$. In addition, if the symmetric part of A_C is negative definite, then $T = I$ and $\tilde{A}_C \equiv A_C$. The bounds (1.61) and (1.62) reduce to (1.53) and (1.55), respectively, in this very special case.

The idea to use a modified DALE for estimating the solution of (1.9) was suggested in [88], where the approach which resulted in the upper external bounds (1.53)-(1.55) was followed. For any stable in discrete-time sense matrix A there exists some positive definite matrix T, such that

$$A^T T^{-2} A - T^{-2} < 0 \Leftrightarrow \tilde{A}^T \tilde{A} < I \Leftrightarrow \sigma_1(\tilde{A}) < 1, \quad \tilde{A} = T^{-1}AT \tag{1.63}$$

in accordance with Theorem 1.7. Pre- and post-multiplication of equation (1.9) by matrix T results in the modified DALE

$$\tilde{A}^T \tilde{P} \tilde{A} - \tilde{P} = -\tilde{Q}, \quad \tilde{P} = TPT, \quad \tilde{Q} = TQT, \quad \sigma_1(\tilde{A}) < 1 \tag{1.64}$$

The scalar and matrix upper bounds for the solution of (1.8) were proposed in [88]:

$$\lambda_1(P) \le \frac{\lambda_1(QT^2)}{\lambda_n^2(T)[1-\sigma_1^2(\tilde{A})]}\ ;\ \text{if}\ \sigma_1(\tilde{A})<1,\ \tilde{A}=T^{-1}AT \tag{1.65}$$

$$tr(P) \le \frac{tr(QT^2)}{\lambda_n^2(T)[1-\sigma_1^2(\tilde{A})]}\ ;\ \text{if}\ \sigma_1(\tilde{A})<1,\ \tilde{A}=T^{-1}AT \tag{1.66}$$

$$tr(P) \le \frac{1}{\lambda_n^2(T)}\sum_{i=1}^{n}\frac{\lambda_i(QT^2)}{[1-\sigma_i^2(\tilde{A})]}\ ;\ \text{if}\ \sigma_1(\tilde{A})<1,\ \tilde{A}=T^{-1}AT \tag{1.67}$$

$$P \le \frac{\lambda_1(QT^2)}{[1-\sigma_1^2(\tilde{A})]}A^TT^{-2}A+Q\ ;\ \text{if}\ \sigma_1(\tilde{A})<1,\ \tilde{A}=T^{-1}AT \tag{1.68}$$

These bounds modify some obtained in [64], [87] respective estimates and actually cover the case when $\sigma_1(A)\ge 1$.

Remark 1.3 Bounds (1.65)-(1.68) generalize some previously respective estimates, since if $\sigma_1(A)<1$, then $T = I$ is a Lyapunov matrix for A. Matrix *A* is assumed to be stable in discrete-time sense and their theoretical validity is guaranteed. All the practical difficulties related with the computation of the external bounds (1.53)-(1.55), (1.61) and (1.62), such as computational complexity, "proper" choice for matrix *T*, resulting in the tightest bound, etc., apply to these bounds, as well.

Under the assumption that *Q* is a strictly positive definite matrix, the lower matrix bounds

$$P \ge \alpha(Q-\alpha^2A^TA)^{1/2},\quad \alpha<\lambda_n^{1/2}[Q(A^TA)^{-1}]\ ;\quad [80] \tag{1.69}$$

$$P \ge \frac{\alpha}{\sigma_1(A)}(Q-\alpha^2I)^{1/2},\quad \alpha<\lambda_n^{1/2}(Q)\ ;\quad [82] \tag{1.70}$$

$$P \ge \frac{1}{\alpha}D^{-1}[D(\alpha Q-Q^{-1})D]^{1/2}D^{-1},\quad D=(AQA^T)^{1/2},\quad \alpha>\frac{1}{\lambda_n^2(Q)}\ ;\quad [83] \tag{1.71}$$

$$P \ge \frac{1}{\alpha\lambda_1^{1/2}(AQA^T)}(\alpha Q-Q^{-1})^{1/2},\quad \alpha>\frac{1}{\lambda_n^2(Q)}\ ;\quad [83] \tag{1.72}$$

for the solution of the CALE (1.2) were proposed and generalized by the same author:

$$P \ge S^{-1}[S(Q-M)S]^{1/2}S^{-1},\quad S=(AM^{-1}A^T)^{1/2},\quad Q>M>0\ ;\quad [83] \tag{1.73}$$

The bounds (1.69)-(1.72) can be obtained from (1.73) by means of a suitable choice for *M*. In the special case when *M* = 0.5*Q*, the bound (1.73) becomes

$$P \geq \frac{1}{2} U^{-1} [UQU]^{1/2} U^{-1}, \quad U = (AQ^{-1}A^T)^{1/2}, \quad Q > 0 \tag{1.74}$$

Another lower matrix bound for the solution of the CALE was proposed in [20]. Under the supposition that Q is a positive definite matrix, the solution P in (1.2) can be bounded from below as follows

$$P \geq \frac{\sqrt{\lambda_n(Q)}}{2} (A^{-1}QA^{-T})^{1/2} \tag{1.75}$$

The upper matrix bound

$$P \leq (A^T + I)P_U(A + I) - A^T P_L A + Q \tag{1.76}$$

was suggested in [83], where P_L and P_U denote lower and upper matrix bound for the solution P of the CALE (1.2), respectively. Assuming that the symmetric part of the coefficient matrix A is negative definite and by making use of the bound for the solution matrix maximal eigenvalue (1.53) with $T = I$, the upper matrix bound in (1.76) may be chosen as

$$P_U = \frac{\lambda_1(Q)}{-2\lambda_1(A_S)} I \tag{1.77}$$

in this case. If the lower matrix bound P_L is given by (1.73), then (1.76) becomes

$$P \leq \frac{\lambda_1(Q)}{-2\lambda_1(A_S)} (A^T + I)(A + I) - A^T P_L A + Q; \text{ if } A_s < 0 \text{ and } Q > 0 \tag{1.78}$$

It is hard to say whether (1.77) is a tighter bound than (1.78) in the general case, but what is sure is that the validity conditions for it combine the conservatism of the lower bounds ($Q > 0$) with the conservatism of the upper bounds ($A_s < 0$). Usually, derivation of upper solution bounds does not require positive definiteness of the right-hand side matrix Q.

Another attempt to overcome the difficulties arising in estimation of the solution of the CARE (1.6) was made in [81], where it is assumed that some positive scalar $\sim$ exists such that the following matrix inequality holds

$$A + A^T < 2\sim BB^T \tag{1.79}$$

Under this supposition the upper matrix bounds

$$P \leq V^{-T} \{ ¨ [(V + I)^T (V + I) + I] + \sim^2 BB^T + Q \} V^{-1} \tag{1.80}$$

$$P \leq V^{-T} [¨ (V + 2I)^T (V + 2I) + 2(\sim^2 BB^T + Q)] V^{-1} \tag{1.81}$$

were proved, where V and ¨ are defined as follows:

$$V = A - \text{~}BB^T - I \;\; , \quad \text{¨} = \frac{\text{ª}_1[V^{-T}(\text{~}^2 BB^T + Q)V^{-1}]}{1 - \text{ª}_1\{V^{-T}[(V+I)^T(V+I)]V^{-1}\}}$$

Note that the scalar ¨ is positive if and only if inequality (1.79) holds. In addition, one more upper matrix bound for the solution of the CARE was suggested in [81]. If the usual for the upper solution bounds validity condition $BB^T > 0$ holds, then

$$P \le E^{-1}[E(A^T SA + Q)E]^{1/2} E^{-1} \tag{1.82}$$

where the positive definite parameter matrix S must be selected to satisfy $BB^T > S^{-1}$ and $E = (BB^T - S^{-1})^{1/2}$.

The validity condition (1.79) is more restrictive than the requirement for stability of matrix $A_C = A - \text{~}BB^T$ and in this sense it is more conservative in comparison with the validity condition under which the upper scalar bounds (1.61) and (1.62) were obtained. Nevertheless, the based on it bounds (1.80) and (1.81) may be viewed as an attempt to reduce the existing conservatism in upper solution bounds for the CARE, since if $BB^T > 0$ the existence of some positive scalar ~ satisfying (1.79) is guaranteed.

The following upper bound for the solution of the CARE was proved in [60]:

$$\text{ª}_1(P) \le \text{¦} = \text{ª}_1(D)\frac{\text{ª}_1[D(Q + K^T K)]}{\text{ª}_n(DM)} \tag{1.83}$$

where K is any m x n matrix stabilizing $A_C = A - BK$. The positive definite matrices D and M are chosen to satisfy the LMI $A_C^T D + DA_C \le -M < 0$.

Under the supposition that Q is a positive definite matrix the lower matrix bound for the solution of the CARE

$$P \ge D^{-1}[D(Q - A^T RA)D]^{1/2} D^{-1}, \quad D = (BB^T + R^{-1})^{1/2}$$

was proposed in [84], where the positive definite parameter matrix R is chosen such that $Q > A^T RA$. The author showed that with appropriate choices of R this bound is tighter and more general than the existing similar bounds.

The available lower bounds for the minimal eigenvalue of the solution of the CARE provide either trivial, or are not valid estimates, if Q is a singular matrix (e.g. see (1.27)-(1.29), (1.75) and (1.82). Consider the slightly modified inequality (1.79)

$$A + A^T < ª_1(BB^T)¦ \tag{1.84}$$

where the positive scalar ¦ is defined in (1.83). Following the approach applied in [81] to derive the upper matrix bounds (1.80), (1.81), and under the supposition that (1.84) is fulfilled, the following lower matrix were reported in [25]:

$$P \geq V^{-T}\{«_1[(V+I)^T(V+I) + (2~ + 1 - ¦ª_1(BB^T))I] + Q\}V^{-1} \tag{1.85}$$

$$P \geq V^{-T}\{«_2[(V+2I)^T(V+2I) + (4 \quad - 2¦ª_1(BB^T))I] + 2Q\}V^{-1} \tag{1.86}$$

$$P \geq W^{-T}\{«_3[(A^TA + (\quad ^2 - \quad ¦ª_1(BB^T))I] + \quad Q\}W^{-1} \tag{1.87}$$

where $V = A - ~I - I, W = A - ¥I$, the positive scalars $~, \quad$ are chosen to satisfy

$$2~ \geq ª_1(BB^T)¦ - 1, ~I > A_S, \qquad > \max[ª_1(A_S), ª_1(BB^T)¦]$$

and the positive scalars $«_1, «_2, «_3$ are defined by

$$«_1 = \frac{ª_n[V^{-T}QV^{-1}]}{1 - ª_n\{V^{-T}[(V+I)^T(V+I) + (2~ + 1 - ¦ª_1(BB^T)I]V^{-1}\}}$$

$$«_2 = \frac{ª_n[V^{-T}QV^{-1}]}{1 - ª_n\{V^{-T}[(V+I)^T(V+I) + (2 \quad + 1 - ¦ª_1(BB^T)I]V^{-1}\}}$$

$$«_3 = \frac{ª_n[W^{-T}QW^{-1}]}{1 - ª_n\{W^{-T}[A^TA + (\quad ^2 - \quad ¦ª_1(BB^T)I]W^{-1}\}}$$

Contrary to the previously obtained lower solution estimates, the bounds (1.85)-(1.87) do not explicitly require positive definiteness of matrix Q.

Motivated by the conservatism of the validity condition (1.79), an approach to derive upper matrix bounds for the positive (semi)-definite solution of the CARE without any additional restrictive conditions imposed on the parameters in (1.6) was suggested in [26]. It resembles to the methods applied in [60], [88] for the derivation of the external upper solution bounds (1.61), (1.62) for CARE and (1.65)-(1.68) for DALE, respectively. Since (A, B) is assumed to be a stabilizable pair, then there always exists some m x n matrix K, such that $A_C = A - BK$ is a stable matrix in the continuous-time sense. Also, in accordance with Theorem 1.7, there exists some nonsingular matrix $D = M^TM$ satisfying the inequality $A_C^TD + DA_C < 0$. The solution matrix can be bounded from above as follows:

$$P \le M^T W^{-T}\{ ° [(W+I)^T (W+I)+I] + M^{-T}(Q+K^T K)M^{-1}\} W^{-1} M \tag{1.88}$$

$$P \le W^{-T}\{ ° [(W+2I)^T (W+2I)+I] + 2M^{-T}(Q+K^T K)M^{-1}\} W^{-1} \tag{1.89}$$

where $W = MA_C M^{-1} - I$ and the positive scalar ° is defined by

$$° = \frac{ª_1[(WM)^{-T}(Q+K^T K)(WM)^{-1}]}{1 - ª_1\{W^{-T}[(W+I)^T (W+I)+I]W^{-1}\}}$$

The validity of both bounds is not restricted by any of the usual additional assumptions, i.e. the estimates in (1.88) and (1.89) are computable whenever a positive (semi)-definite solution for the CARE exists.

A similar approach was applied to derive non-conservative upper solution bounds for the positive (semi)-definite solution of the DARE in [27]. If there exists some *m* x *n* matrix *K* such that

$$A_C^T A_C < I, A_C = A - BK \tag{1.90}$$

then the solution of (1.14) has the following upper bound

$$P \le ©A_C^T A_C + Q + K^T K, \quad © = \frac{ª_1(Q+KK)}{1-±_1^2(A_c)} \tag{1.91}$$

Realizing that the condition (1.90) is rather conservative, the authors suggested a relaxed solution upper matrix bound for the DARE:

$$P \le «A_C^T D^T D A_C + Q + K^T K, \quad « = \frac{ª_1[D^{-T}(Q+K^T K)D^{-1}]}{1-±_1^2[D(A_c)D^{-1}]} \tag{1.92}$$

where matrix D is chosen to satisfy the inequality $A_C^T M A_C < M, M = D^T D > 0$, which represents the validity constraint for this estimate. Since (A, B) is a stabilizable pair, in view of Theorem 1.7, this condition can always be met, which guarantees the validity of (1.92).

For any positive scalar ~ the CALE can be equivalently represented as a modified DALE

$$\tilde{A}^T P \tilde{A} - P = -\tilde{Q}, \quad \tilde{A} = (A - {\sim} I)^{-1}(A + {\sim} I), \quad \tilde{Q} = 2{\sim}(A - {\sim} I)^{-T} Q (A - {\sim} I)^{-1} \tag{1.93}$$

Therefore, (1.2) and (1.93) have one and the same solution *P*. From this equation it follows that $\tilde{A}$ is stable in the discrete-time sense if and only if *A* is stable in the continuous-time sense. Moreover, one can easily verify that

$$A_S < 0 \Leftrightarrow \tilde{A}^T \tilde{A} < I \tag{1.94}$$

First of all, note that

$$P \geq \tilde{Q}, \forall \mu > 0 \qquad (1.95)$$

The matrix bound (1.95) for P in (1.2) is completely independent from the usual for lower estimates additional assumption that Q is a positive definite matrix. These facts were used to derive lower and upper bounds for the solution of the CALE in [79], where for $\mu = 1$ in (1.93) the following estimates were proved

$$P \geq \delta \tilde{A}^T \tilde{A} + \tilde{Q}, \qquad \delta = \frac{\lambda_n(\tilde{Q})}{1 - \lambda_n(\tilde{A}^T \tilde{A})} \qquad (1.96)$$

$$P \geq L = \tilde{A}^T \tilde{Q} \tilde{A} + \tilde{Q} \qquad (1.97)$$

This bound (1.96) is tighter than (1.95) only if Q is a strictly positive definite matrix. If Q is a singular matrix, the same refers to $\tilde{Q}$ and then $\delta = 0$, which means that the two bounds provide the same lower estimate. The estimate L in (1.97) is always applicable and among these three bounds is the tightest.

Under the supposition that the symmetric part of the coefficient matrix A is negative definite, the following based on the representation (1.93) upper bounds for the solution of the CALE were obtained in [79]:

$$P \leq U = \gamma \tilde{A}^T \tilde{A} + \tilde{Q}, \quad \gamma = \frac{\lambda_1(\tilde{Q})}{1 - \lambda_1(\tilde{A}^T \tilde{A})}; \text{ if } A_s < 0 \qquad (1.98)$$

$$P \leq (A^T + I)U(A^T + I) - ALA + Q; \text{ if } A_s < 0 \qquad (1.99)$$

where matrix L is defined in (1.97).

If Q is a positive definite matrix a lower matrix bound for the solution of the CARE (1.6)

$$P \geq [\alpha Q - \alpha^2 A R^{-1} A^T + \frac{1}{4} \beta^2 \lambda_n(R) I]^{1/2}, \; R = I - \alpha B B^T \qquad (1.100)$$

was suggested in [143]. The positive scalars α, β are defined as follows:

$$\alpha \leq \min\{\frac{\beta}{2\sigma_1(A) + \beta \lambda_1(BB^T)}, \frac{1}{\sigma_1(BB^T + A^T Q^{-1} A)}\}, \; \beta = \frac{1}{\lambda_n(A_S Q^{-1}) + [\lambda_n^2(A_S Q^{-1}) + \lambda_1(BB^T Q^{-1})]^{1/2}}$$

The authors claim that (1.100) is tighter than a previously obtained in [20] similar solution estimate

$$P \geq [\alpha Q - \alpha^2 A R^{-1} A^T)]^{1/2}, \; R = I - \alpha B B^T \; \alpha \leq \frac{1}{\sigma_1^2(BB^T + A^T Q^{-1} A)} \qquad (1.101)$$

Assuming that Q is a positive definite matrix the authors proposed the upper matrix bound for the solution of the CARE

$$P \leq \{£Q - £^2 AR^{-1}A^T + ª_1(R)[² + £±_1(AR^{-1})]^2 I\}^{1/2},\ R = I - £BB^T \qquad (1.102)$$

in [143]. The positive scalar £ is chosen to satisfy the inequality in (1.101) and ² is defined as follows:

$$² = \frac{1}{¯}\left\{« + \left\langle «^2 - ¯\left[£^2ª_n(AR^{-1}A^T) - £ª_1(Q) - \frac{«^2}{1-¯}\right]\right\rangle^{1/2}\right\},\quad ¯ = 1 - ª_1(R), « = £ª_1(R)±_1(AR^{-1})$$

Although not explicitly said, the upper bound (1.102) is valid only if $BB^T > 0$, which becomes clear from the following simple fact. Suppose that BB^T is a singular matrix, i.e. $ª_n(BB^T) = 0$. This means that

$$¯ = 1 - ª_1(R) = 1 - ª_1(I - £BB^T) = 1 - 1 + £ª_n(BB^T) = £ª_n(BB^T) = 0, \forall £$$

In other words, the inverse of ¯ does not exist. The same refers to ² and the bound (1.102), as well.

A similar approach is applied in [91]. Under the suppositions that $Q > 0$ (for the lower bound) and $BB^T > 0$ (for the upper bound) it has been found that the positive definite solution of the CARE satisfies the matrix inequalities

$$P \geq [£Q - £^2(V+I)^T R^{-1}(V+I) + £^2~^2 + \frac{1}{4}©^2ª_n(R)I]^{1/2} + £~I,\ R = I - £BB^T \qquad (1.103)$$

where $A + A^T < 2~I, V = A - (~ + 1)I$, and © is defined by

$$© = \frac{ª_1(BB^TQ^{-1})}{-ª_n(A_SQ^{-1}) + [ª_n^2(A_SQ^{-1}) + ª_1(BB^TQ^{-1})]^{1/2}}$$

the positive scalar £ is selected to satisfy the inequality

$$£ \leq \min\{\frac{©}{2±_1(V+I) + ¦ª_1(BB^T)}, \frac{1}{±_1[BB^T + (V+I)^TQ^{-1}(V+I)]}, \frac{©}{~}\}$$

and

$$P \leq \{(1-¯)[² + £±_1[R^{-1}(V+I)]]^2 - £^2(V+I)^TR^{-1}(V+I) + £^2~^2I + £Q\}^{1/2} + £~I \qquad (1.104)$$

where ¯ and © are the same as in (1.102) and (1.103), respectively, the positive scalar ~ satisfies the inequality used to derive bound (1.103), the positive scalar £ is selected as

$$£ \le \min\{\frac{1}{±_1[BB^T + (V+I)^T Q^{-1}(V+I)]}, \frac{©}{\sim}\},$$

and ² is given by

$$² = \frac{1}{¯}\left\{ « + «^2 + ¯ \left\langle \frac{«^2}{1-¯} - ª_n\left[£^2 (V+I)^T R^{-1}(V+I)\right] + £ª_1(Q) \right\rangle \right\}^{1/2}$$

where $« = £(1-¯)±_1[R^{-1}(V+I)] + £\sim$.

The trace estimation problem for the solution of the DARE was also investigated in [23]. Without imposing any additional restrictions, the authors derived the following lower trace bounds for the unique positive definite solution P:

$$tr(P) \ge \sum_{i=n-r_a+1}^{n} \frac{±^2_{n-i+1}(A)ª_i(Q)}{ª_i(Q)ª_1(BB^T)}, \quad r_a = rank(A) \le n \qquad (1.105)$$

$$tr(P) \ge \sum_{i=n-r_q+1}^{n} \frac{ª_{n-i+1}(Q)±^2_i(A)}{ª_i(Q)ª_1(BB^T)}, \quad r_q = rank(Q) \le n \qquad (1.106)$$

In addition, if $r_a = r_q = n$, then

$$tr(P) \ge \frac{¯ + [¯^2 + 4ª_1(BB^T)tr(Q)]^{1/2}}{2ª_1(BB^T)}, \quad ¯ = ±^2_n(A) + ª_1(BB^T)ª_n(Q) - 1$$

in this special case. The solution trace can be bounded from above as:

$$tr(P) \le \frac{n\{¯ + [¯^2 + 4\frac{¦}{n}ª_r(BB^T)]^{1/2}}{2ª_r(BB^T)}; \text{ if } AA^T < I \qquad (1.107)$$

where $¯ = ±^2_1(A) + ª_r(BB^T)ª_1(Q) - 1, \; ¦ = tr(Q) + \min\{\sim, \; \}$,

$$\sim = (n-r)ª_r(BB^T)\left[\frac{±_1(A)ª_1(Q)}{1-±^2_1(A)}\right]^2, \qquad = ª_r(BB^T)\left[\frac{±_1(A)\sum_{i=1}^{n-r}ª_i(Q)}{1-±^2_1(A)}\right]^2$$

and $r \le n$ denotes the rank of matrix B. If $r = n$, i.e. $BB^T > 0$, then $¦ = tr(Q)$ and (1.107) becomes

$$tr(P) \le \frac{n\{¯ + [¯^2 + 4\frac{tr(Q)}{n}ª_n(BB^T)]^{1/2}}{2ª_n(BB^T)}; \text{ if } AA^T < I \text{ and } BB^T > 0 \qquad (1.108)$$

where $\overline{\ } = \pm_1^2(A) + \text{ª}_n(BB^T)\text{ª}_1(Q) - 1$. In particular, the maximal solution eigenvalue satisfies the inequality

$$\text{ª}_1(P) \leq \frac{\overline{\ } + [\overline{\ }^2 + 4\text{ª}_n(BB^T)\text{ª}_1(BB^T)]^{1/2}}{2\text{ª}_n(BB^T)}; \text{ if } AA^T < I \text{ and } BB^T > 0 \qquad (1.109)$$

A brief analysis of the bounds obtained during the second period, i.e. after 1995, reveals the following trends in the solution estimation process:

1. Realizing the embarrassing fact that all upper bounds for the CALE and the DALE solutions depend crucially on the conditions $A_s < 0$ (for the continuous-time case) and $AA^T < I$ (for the discrete-time case), several authors suggested the application of suitable nonsingular transformations for the coefficient matrix *A*, which guarantees the bounds validity for the solutions of the obtained modified equations. This approach gave rise to the so called external bounds (see bounds (1.53)-(1.55), (1.65)-(1.68), (1.83)) which include some additional computational procedure, e.g. LMI solution. Although such bounds are theoretically always valid, they demonstrate two main shortcomings:

(i) the additional computational burden may be significant and even comparable with the one needed for the direct solution of the respective equation,

(ii) there does not exist a systematic way to select the "best" transformation which results in the tightest bound.

2. Several attempts to overcome the difficulties associated with the estimation for the CARE and the DARE solutions (requirements for strict positive definiteness of matrices *Q* and BB^T) were made. E.g. using the fact that the pair (*A*, *B*) is assumed to be stabilizable, an approach to bind the solution of the CARE from above by the solution of a respective CALE was applied to derive the bounds (1.61), (1.62) and (1.92). Some more, or less conservative conditions under which the bounds (1.80), (1.81), (1.85)-(1.87) hold were obtained, as well.

3. With time, the bounds became more and more complex, often including several scalar or matrix parameter which must be selected in accordance with some additional requirements (e.g., see bounds (1.71)-(1.73), (1.80)-(1.82), (1.84), (1.85)-(1.87), (1.100)-(1.103)).

4. According to the common opinion it is very hard to compare different respective bounds, especially the scalar ones, with respect to tightness. No much attention has been paid to the solution of the important problems of bounds sharpness and improvement.

Motivated by the significance of the estimation problem for the four considered algebraic equations and the various open problems in this research field, we consider the following main trends:

(i) extension of the set of stable coefficient matrices *A* for which there exist valid upper solution bounds for the CALE and the DALE solutions,

(ii) application of this extended set for deriving respective bounds for the CARE and the DARE solutions,

(iii) improvement of some existing lower and upper solution bounds in sense of tightness.

CHAPTER TWO

BOUNDS FOR THE CONTINUOUS-TIME EQUATIONS

2.1 THE SINGULAR VALUE DECOMPOSITION APPROACH

Let *A* be an arbitrary *n* x *n* matrix. There exist orthogonal (unitary) matrices *U* and *V*, such that *A* can be represented as follows [49]:

$$A = U\Sigma V^T = (U\Sigma U^T)(UV^T) = (UV^T)(V\Sigma V^T) = R_1F = FR_2 \qquad (2.1)$$

where Σ is a diagonal matrix containing the *n* singular values of *A*, $F = UV^T$ and R_1, R_2 are matrices defined as

$$R_1 = (AA^T)^{1/2}, \quad R_2 = (A^T A)^{1/2} \tag{2.2}$$

Therefore, any square matrix *A* can be represented as a product of a symmetric matrix and an orthogonal matrix *F*. The eigenvalues of this symmetric matrix are exactly the singular values of *A*. In addition, if *A* is nonsingular, then R_1, R_2 are positive definite matrices.

Define the following matrix sets:

$$\boldsymbol{H} \equiv \{A, A \in \boldsymbol{R}_{\boldsymbol{n}} : \det(A - \lambda I) = 0 \Rightarrow \operatorname{Re} \lambda < 0\}$$

$$\boldsymbol{H}^- \equiv \{A, A \in \boldsymbol{R}_{\boldsymbol{n}} : A_S < 0\}$$

$$\tilde{\boldsymbol{H}} \equiv \{A, A \in \boldsymbol{R}_{\boldsymbol{n}} : F \in \boldsymbol{H}\}$$

i.e. $\boldsymbol{H}$ is the set of stable matrices, $\boldsymbol{H}^-$ is the set of matrices with negative definite symmetric parts and $\tilde{\boldsymbol{H}}$ is the set of matrices with stable orthogonal parts *F* in (2.1).

Theorem 2.1. Consider the singular value decomposition of matrix A (2.1)-(2.2). Then,

1. $\boldsymbol{H}^- \subseteq \tilde{\boldsymbol{H}}$,
2. $A \in \boldsymbol{H}^-$ only if R_1^{-1} and R_2 are Lyapunov matrices for *A*.
3. $A \in \tilde{\boldsymbol{H}}$ if and only if R_1^{-1} and R_2 are Lyapunov matrices for *A*.

Proof. Let $A \in \boldsymbol{H}^-$ i.e.

$$0 > 2A_S = F^T R_1 + R_1 F = R_2 F^T + F R_2 \Rightarrow F \in \boldsymbol{H} \Rightarrow A \in \tilde{\boldsymbol{H}}$$

in accordance with Theorem 1.7. This proves the first statement.

If $A \in \boldsymbol{H}^-$, then *F* is a stable matrix, but being unitary, it is normal, as well, and therefore can be diagonalized via a unitary matrix transformation [49], i.e.

$$W^T F W = \Lambda, \quad \Lambda = diag.\{\lambda_i(F)\}, \operatorname{Re} \lambda_i(F) < 0, i = 1,..,n$$

It follows that $F \in \boldsymbol{H} \Leftrightarrow F \in \boldsymbol{H}^-$. Consider the matrix inequalities:

$$A^T R_1^{-1} + R_1^{-1} A = 2F_S < 0, \quad A^T R_2 + R_2 A = 2R_2 F_S R_2 < 0 \tag{2.3}$$

Having in mind Definition 1.3, it follows that R_1^{-1} and R_2 are Lyapunov matrices for *A*, in this case. This proves the second statement of the Theorem.

Having in mind that

$$A^T R_1^{-1} + R_1^{-1} A = 2F_S \quad A^T R_2 + R_2 A = 2R_2 F_S R_2$$

statement (3) is obvious.

Example 2.1 Consider the following example:

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & ® \end{bmatrix}, \quad ª_1(A) = -1,\ ª_{2,3}(A) = \frac{® - 1 \pm \sqrt{(® - 1)^2 - 4(1 - ®)}}{2}$$

Matrix A is stable for all $® < 1$, while $A_S < 0$ for $® < -0.25$. It is interesting to see how changes in the parameter ® influence the maximal eigenvalues of the symmetric parts of *A* and *F*. The results are given in Table 2.1, where $\sim = \max \mathrm{Re}\, ª_i(A),\ i = 1, 2, 3$.

It is clear that the based on the condition for negative definiteness of the symmetric part of the coefficient matrix *A* available upper bounds for the maximal eigenvalue, the trace and the solution matrix for CALE are inapplicable for all $® \in [-0.25, 1)$ in this case. At the same time, matrix F_S is strictly negative definite. We shall see how this important fact can help the derivation of valid solution upper bounds.

®	~	$ª_1(A_S)$	$ª_1(F_S)$
0.9	-0.05	1.0236	-0.0045
0.5	-0.25	0.6514	-0.1774
0	-0.50	0.2071	-0.3711
-0.25	-0.625	0.0000	-0.4538
-0.5	-0.75	-0.191	-0.5262
-1	-1	-0.5	-0.6421
-2	-1	-0.7929	-0.7869

Table 2.1 Dependence of $\sim, ª_1(A_S), ª_1(F_S)$ on ®

2.2 BOUNDS FOR THE CALE SOLUTION

2.2.1 TRACE BOUNDS

The **s**ignificance of the singular value decomposition of the coefficient matrix *A* for estimation purposes was firstly investigated in [122], where new upper bound for the solution trace of the CALE was proposed. Before presenting this result, recall some well known properties of the trace operator (sum of eigenvalues = sum of diagonal entries). For arbitrary square matrices X , Y, and a symmetric matrix Z, one has:

$$tr(X) = tr(X^T),\ tr(X) + tr(Y) = tr(X + Y),\ tr(XY) = tr(YX),\ tr(XZ) = tr(X_S Z)$$

Theorem 2.2 Let $A \in \tilde{\boldsymbol{H}}$. Then the trace of the CALE (1.2) solution has the upper bound:

$$tr(P) \leq u_1 = \frac{tr(R_1 Q)}{-2\lambda_1(R_1 F_S R_1)} \tag{2.4}$$

Proof. Having in mind (2.1) and (2.2) the CALE can be rewritten as

$$F^T R_1 P + P R_1 F = -Q \tag{2.5}$$

Pre-multiplication of (2.5) by matrix R_1 and application of the trace operator to both sides of the resultant matrix equality leads to:

$$R_1 F^T R_1 P + R_1 P R_1 F = -R_1 Q \Rightarrow -tr(R_1 Q) = tr(R_1 F^T R_1 P + P R_1 F R_1) = 2tr(R_1 F_S R_1 P) \leq 2\lambda_1(R_1 F_S R_1) tr(P)$$

By assumption matrix $A \in \tilde{\boldsymbol{H}}$ and therefore

$$F \in \boldsymbol{H} \Leftrightarrow F \in \boldsymbol{H}^- \Leftrightarrow \lambda_1(F_S) < 0 \Leftrightarrow \lambda_1(R_1 F_S R_1) < 0$$

which proves the upper trace bound (2.4).

Later on, this result was extended in [124].

Theorem 2.3 Let $A \in \tilde{\boldsymbol{H}}$. Then the trace of the CALE (1.2) solution has the upper bound

$$tr(P) \leq u_2 = \frac{tr(R_2^{-1} Q)}{-2\lambda_1(F_S)} \tag{2.6}$$

Proof. Having in mind the singular value decomposition of matrix A in (2.1) and (2.2) consider the CALE (1.2) rewritten as

$$R_2 F^T P + P F R_2 = -Q \tag{2.7}$$

Pre-multiplication of (2.7) by the inverse of matrix R_2 and application of the trace operator to both sides of the resultant matrix equality leads to

$$F^T P + R_2^{-1} P F R_2 = -R_2^{-1} Q \Rightarrow -tr(R_2^{-1} Q) = tr(F^T P + PF) = 2tr(F_S P) \leq 2\lambda_1(F_S) tr(P)$$

By assumption matrix $A \in \tilde{\boldsymbol{H}}$ and therefore $F \in \boldsymbol{H} \Leftrightarrow F \in \boldsymbol{H}^- \Leftrightarrow \lambda_1(F_S) < 0$, which proves the trace inequality in (2.6).

If $A \in \boldsymbol{H}^-$, bound (1.23) is considered to be the tightest upper trace estimate for the CALE solution. This fact was used in [122] to get a modified less restrictive validity condition.

Theorem 2.4 Suppose that $A \in \tilde{\boldsymbol{H}}$. Then the trace of the CALE solution has the upper bound

$$tr(P) \le u_3 = \min\left(\frac{1}{\alpha_n(A)}\sum_{i=1}^{n}\frac{\lambda_i(R_1Q)}{-2\lambda_1(R_1F_S)},\ \alpha_1(A)\sum_{i=1}^{n}\frac{\lambda_i(R_2^{-1}Q)}{-2\lambda_1(R_2F_S)}\right) \qquad (2.8)$$

Proof. Consider the CALE (2.5). Pre- and post-multiplication by $R_1^{1/2}$ results in the following modified CALE equation

$$(R_1^{1/2}F^TR_1^{1/2})(R_1^{1/2}PR_1^{1/2}) + (R_1^{1/2}PR_1^{1/2})(R_1^{1/2}FR_1^{1/2}) = -R_1^{1/2}QR_1^{1/2} \qquad (2.9)$$

Denoting $\tilde{F} = R_1^{1/2}FR_1^{1/2}$, $\tilde{P} = R_1^{1/2}PR_1^{1/2}$ and $\tilde{Q} = R_1^{1/2}QR_1^{1/2}$, (2.9) can be rewritten more compactly as

$$\tilde{F}^T\tilde{P} + \tilde{P}\tilde{F} = -\tilde{Q} \qquad (2.10)$$

If the supposition $A \in \tilde{\boldsymbol{H}}$ holds, then $\tilde{F} \in \boldsymbol{H}^-$ and (1.23) can be applied to get the upper bound for the solution matrix of (2.10), i.e.

$$tr(\tilde{P}) \le \sum_{i=1}^{n}\frac{\lambda_i(\tilde{Q})}{-2\lambda_i(\tilde{F}_S)}$$

Using the trace inequality $tr(\tilde{P}) = tr(R_1P) \ge \lambda_n(R_1)tr(P) = \alpha_n(A)tr(P)$, one finally gets the trace estimate for *P*:

$$tr(P) \le \frac{1}{\alpha_n(A)}\sum_{i=1}^{n}\frac{\lambda_i(\tilde{Q})}{-2\lambda_i(\tilde{F}_S)} = \frac{1}{\alpha_n(A)}\sum_{i=1}^{n}\frac{\lambda_i(R_1Q)}{-2\lambda_1(R_1F_S)} \qquad (2.11)$$

Consider the CALE (2.7) pre- and post-multiplied by $R_2^{-1/2}$:

$$(R_2^{1/2}F^TR_2^{1/2})(R_2^{-1/2}PR_2^{-1/2}) + (R_2^{-1/2}PR_2^{-1/2})(R_2^{1/2}FR_2^{1/2}) = -R_2^{-1/2}QR_2^{-1/2} \qquad (2.12)$$

Denote $\overline{F} = R_2^{1/2}FR_2^{1/2}$ $\overline{P} = R_2^{-1/2}PR_2^{-1/2}$ $\overline{Q} = R_2^{-1/2}QR_2^{-1/2}$. This helps to put (2.12) in the form

$$\overline{F}^T\overline{P} + \overline{P}\overline{F} = -\overline{Q} \qquad (2.13)$$

If the supposition $A \in \tilde{\boldsymbol{H}}$ holds, then $\overline{F} \in \boldsymbol{H}^-$ and bound (1.23) can be applied again to get the upper bound for the solution matrix of (2.13), i.e.

$$tr(\overline{P}) \le \sum_{i=1}^{n}\frac{\lambda_i(\overline{Q})}{-2\lambda_i(\overline{F}_S)}$$

The trace of the solution matrix in (2.13) can be bounded from below as follows:

$$tr(\overline{P}) = tr(R_2^{-1}P) \geq \lambda_n(R_2^{-1})tr(P) = \frac{1}{\lambda_1(R)}tr(P) = \frac{1}{\alpha_1(A)}tr(P)$$

This leads to the next upper bound for the solution of the CALE (1.2)

$$tr(P) \leq \alpha_1(A)\sum_{i=1}^{n}\frac{\lambda_i(\overline{Q})}{-2\lambda_i(\overline{F}_S)} = \alpha_1(A)\sum_{i=1}^{n}\frac{\lambda_i(R_2^{-1}Q)}{-2\lambda_1(R_2F_S)} \tag{2.14}$$

Having in mind (2.11) and (2.14), the bound in (2.8) follows.

Corollary 2.1 If $A \in \widetilde{\boldsymbol{H}}$, then the trace of the solution of the CALE (1.2) has the upper bound

$$tr(P) \leq t_{U1} = \min(u_1, u_2, u_3) \tag{2.15}$$

where u_1, u_2, u_3 are bounds defined in (2.4), (2.6), (2.8), respectively.

The upper trace bound (2.15) was reported in [121].

Remark 2.1 The upper trace bound (2.15) can be computed under less restrictive conditions imposed on the coefficient matrix A in (1.2) and in this sense it is less conservative in comparison with all known similar bounds.

Provided that the CALE (1.2) has positive (semi)-definite solution, lower trace bounds for it can always be derived. Application of the singular value decomposition approach results in the derivation of some lower trace estimates, as well.

Theorem 2.5 The trace of the positive (semi)-definite solution P of the CALE (1.2) has the following lower bounds:

$$tr(P) \geq t_{L1} = \max(l_1, l_2)\,,\ \ l_1 = \frac{tr(R_1Q)}{-2\lambda_n(R_1F_SR_1)},\ \ l_2 = \frac{tr(R_2^{-1}Q)}{-2\lambda_n(F_S)} \tag{2.16}$$

Proof. Pre-multiplication of (2.5) by matrix R_1 and application of the trace operator to both sides of the resultant matrix equality results in

$$R_1F^TR_1P + R_1PR_1F = -R_1Q \Rightarrow -tr(R_1Q) = tr(R_1F^TR_1P + PR_1FR_1) = 2tr(R_1F_SR_1P) \geq 2\lambda_n(R_1F_SR_1)tr(P)$$

and the bound l_1 follows.

Pre-multiplication of (2.7) by the inverse of matrix R_2 and application of the trace operator to both sides of the resultant matrix equality leads to

$$F^TP + R_2^{-1}PFR_2 = -R_2^{-1}Q \Rightarrow -tr(R_2^{-1}Q) = tr(F^TP + PF) = 2tr(F_SP) \geq 2\lambda_n(F_S)tr(P)$$

which proves bound l_2 and the statement of the Theorem, as well.

2.2.2 BOUNDS FOR THE EXTREMAL EIGENVALUES

The singular value decomposition approach can be applied to derive some upper and lower bounds for the maximal and minimal eigenvalues of the CALE solution.

Theorem 2.6 Suppose that $A \in \tilde{\boldsymbol{H}}$. Then the maximal solution eigenvalue for the CALE has the following upper bound:

$$\lambda_1(P) \le e_{U1} = \min\left(\frac{\lambda_1(-QF_S^{-1})}{2\sigma_n(A)},\ \frac{\sigma_1(A)\lambda_1[-Q(R_2F_SR_2)^{-1}]}{2}\right) \tag{2.17}$$

Proof. Consider the modified CALE (2.9). Let x be an eigenvector for matrix $R_1^{1/2}FR_1^{1/2}$ corresponding to its largest eigenvalue, i.e. $R_1^{1/2}FR_1^{1/2}x = \lambda_1 x$. Then,

$$x^T[(R_1^{1/2}F^TR_1^{1/2})(R_1^{1/2}PR_1^{1/2}) + (R_1^{1/2}PR_1^{1/2})(R_1^{1/2}FR_1^{1/2})]x = 2\lambda_1 x^T(R_1^{1/2}F_SR_1^{1/2})x = -x^TR_1^{1/2}QR_1^{1/2}x$$

Since $A \in \tilde{\boldsymbol{H}}$ by assumption, it follows that $R_1^{1/2}F_SR_1^{1/2} < 0$. Denote $(-F_S)^{1/2}R_1^{1/2}x = y$, to get the following equality for the maximal solution eigenvalue of (2.9)

$$\lambda_1 = \frac{y^T[(-F_S)^{-1/2}Q(-F_S)^{-1/2}]y}{2y^Ty} \le \frac{\lambda_1(-QF_S^{-1})}{2}$$

which can be bounded from below as

$$\lambda_1 = \lambda_1(R_1^{1/2}F_SR_1^{1/2}) \ge \lambda_n(R_1)\lambda_1(P) = \sigma_n(A)\lambda_1(P)$$

This proves the first upper eigenvalue bound in (2.17).

Consider now the modified CALE (2.12). Let x be an eigenvector for matrix $R_2^{-1/2}PR_2^{-1/2}$ corresponding to its largest eigenvalue, i.e. $R_2^{-1/2}PR_2^{-1/2}x = \lambda_1 x$ and consider the scalar equality

$$x^T[(R_2^{1/2}F^TR_2^{1/2})(R_2^{-1/2}PR_2^{-1/2}) + (R_2^{-1/2}PR_2^{-1/2})(R_2^{1/2}FR_2^{1/2})]x = 2\lambda_1 x^T(R_2^{1/2}F_SR_2^{1/2})x = -R_2^{-1/2}QR_2^{-1/2}$$

It is assumed that $A \in \tilde{\boldsymbol{H}}$, therefore $R_2^{1/2}F_SR_2^{1/2} < 0$. Using the notation $(-F_S)^{1/2}R_2^{1/2}x = y$, one gets

$$\lambda_1 = \frac{y^T[(-F_S)^{-1/2}R_2^{-1}QR_2^{-1}(-F_S)^{-1/2}]y}{2y^Ty} \le \frac{\lambda_1[-Q(R_2F_sR_2)^{-1}]}{2}$$

The maximal eigenvalue of the solution of (2.12) can be evaluated as:

$$\lambda_1 = \lambda_1(R_2^{-1/2}PR_2^{-1/2}) \ge \lambda_n(R_2^{-1})\lambda_1(P) = \frac{\lambda_1(P)}{\lambda_1(R)} = \frac{\lambda_1(P)}{\sigma_1(A)}$$

and the second upper bound in (2.17) follows. This proves the statement of the Theorem.

Remark 2.2 The upper eigenvalue bound (2.17) can be computed under less restrictive conditions imposed on the coefficient matrix A in (1.2) and in this sense it is less conservative in comparison with all known similar bounds.

The same approach can be applied to get lower bounds for the minimal eigenvalue of the CALE solution matrix.

Theorem 2.7 Suppose that Q in (1.2) is a positive definite matrix. Then the minimal eigenvalue of P has the following lower bound:

$$\lambda_n(P) \geq e_{L1} = \max\left(\frac{1}{-2\sigma_1(A)\lambda_n(F_S Q^{-1})},\ \frac{\sigma_n(A)}{-2\lambda_n(R_2 F_S R_2 Q^{-1})}\right) \qquad (2.18)$$

Proof. Consider the modified CALE (2.9). Let x be an eigenvector for matrix $R_1^{1/2} P R_1^{1/2}$ corresponding to its minimal eigenvalue, i.e. $R_1^{1/2} P R_1^{1/2} x = \lambda_n x$. Then,

$$x^T[(R_1^{1/2} F^T R_1^{1/2})(R_1^{1/2} P R_1^{1/2}) + (R_1^{1/2} P R_1^{1/2})(R_1^{1/2} F R_1^{1/2})]x = 2\lambda_n x^T (R_1^{1/2} F_S R_1^{1/2}) x = -x^T R_1^{1/2} Q R_1^{1/2} x$$

By assumption $Q > 0$, i.e. $Q^{-1/2}$ exists. Let

$$Q^{1/2} R_1^{1/2} x = y \Rightarrow x = R_1^{-1/2} Q^{-1/2} y \Rightarrow 2\lambda_n y^T Q^{-1/2} F_S Q^{-1/2} y = -y^T y$$

or,

$$-1 = 2\lambda_n \frac{y^T Q^{-1/2} F_S Q^{-1/2} y}{y^T y} \geq 2\lambda_n \lambda_n(F_S Q^{-1}) \Rightarrow \lambda_n \geq \frac{1}{-2\lambda_n(F_S Q^{-1})}$$

The minimal eigenvalue λ_n of $R_1^{1/2} P R_1^{1/2}$ can be additionally bounded from above as $\lambda_n \leq \lambda_1(R_1)\lambda_n(P) = \sigma_1(A)\lambda_n(P)$ and the first bound in (2.18) is obtained.

Consider the modified CALE (2.12) Let x be an eigenvector for matrix $R_2^{-1/2} P R_2^{-1/2}$ corresponding to its minimal eigenvalue, i.e. $R_2^{-1/2} P R_2^{-1/2} x = \lambda_n x$ and consider the scalar equality

$$x^T[(R_2^{1/2} F^T R_2^{1/2})(R_2^{-1/2} P R_2^{-1/2}) + (R_2^{-1/2} P R_2^{-1/2})(R_2^{1/2} F R_2^{1/2})]x = 2\lambda_n x^T (R_2^{1/2} F_S R_2^{1/2}) x = -R_2^{-1/2} Q R_2^{-1/2}$$

Define the vector $Q^{1/2} R_2^{-1/2} x = y$ and having in mind that $Q > 0$ by assumption, one gets

$$-1 = 2\lambda_n \frac{y^T Q^{-1/2} R_2 F_S R_2 Q^{-1/2} y}{y^T y} \geq 2\lambda_n \lambda_n(R_2 F_S R_2 Q^{-1}) \Rightarrow \lambda_n \geq \frac{1}{-2\lambda_n(R_2 F_S R_2 Q^{-1})}$$

Using the inequality $\lambda_n \le \lambda_1(R_2^{-1})\lambda_n(P) = \dfrac{\lambda_n(P)}{\lambda_n(R_2)} = \dfrac{\lambda_n(P)}{\sigma_n(A)}$, the bound in (2.18) is proved.

As all available similar estimates, the lower eigenvalue bound in (2.18) is dependent on the condition $Q > 0$. The next result is an attempt to relax this restrictive requirement.

Theorem 2.8 Suppose that $\tilde{Q} = Q + A^T QA > 0$. Then the minimal eigenvalue of the CALE solution has the following lower bound

$$\lambda_n(P) \ge e_{L2} = \frac{1}{-2\lambda_n\{[(A^T A + I)A]_S \tilde{Q}^{-1}\}} \tag{2.19}$$

Proof. Consider (1.2) multiplied by A^{-T} and A^{-1} from the left and from the right, respectively. i.e.

$$A^{-T}P + PA^{-1} = -A^{-T}QA^{-1}.$$

Then P is the unique solution of the CALE

$$(A^T + A^{-T})P + P(A + A^{-1}) = -\overline{Q} = -A^{-T}\tilde{Q}A^{-1}, \quad \overline{Q} > 0 \tag{2.20}$$

Denote $\overline{A} = A + A^{-1}$. Application of the valid in this case bound (1.17) leads to the estimate

$$\lambda_n(P) \ge \frac{1}{-2\lambda_n(\overline{A}_S\overline{Q}^{-1})} = \frac{1}{-\lambda_n[(A^T + A + A^{-T} + A^{-1})A\tilde{Q}^{-1}A^T]} = \frac{1}{-\lambda_n[(A^{2T}A + A^T A^2 + A^T + A)\tilde{Q}^{-1}]}$$

which proves the bound in (2.19).

Application of the singular value decomposition approach helps to derive the next lower eigenvalue bound.

Theorem 2.9 Suppose that $\tilde{Q} = Q + A^T QA > 0$. The minimal eigenvalue of the solution of the CALE (1.2) has the following lower bound

$$\lambda_n(P) \ge e_{L3} = \max\left(\frac{1}{-2\sigma_1(\overline{A})\lambda_n(\overline{A}^T\overline{F}_S\overline{A}\tilde{Q}^{-1})}, \frac{\sigma_n(\overline{A})}{-2\lambda_n(\overline{A}^T\overline{R}_2\overline{F}_S\overline{R}_2\overline{A}\tilde{Q}^{-1})}\right) \tag{2.21}$$

where $\overline{A} = A + A^{-1} = \overline{F}\,\overline{R}_2$, $\overline{R}_2 = (\overline{A}^T\overline{A})^{1/2}$ is the singular value decomposition of matrix $\overline{A}$.

Proof. Having in mind that the bounds in (2.21) are based on the applied to the solution of the CALE (2.20) bounds in (2.18), the proof follows easily and is omitted.

Remark 2.3 The lower bounds for the minimal eigenvalue of the CALE (1.2) solution relax the condition ($Q > 0$) under which available similar estimates are valid. It is clear that if Q is a positive definite matrix then $\tilde{Q} = Q + A^T QA > 0$. Up to our best knowledge the bounds (2.19) and (2.21) are the first ones which are computable in some cases when the right-hand side matrix in (1.2) is not strictly positive definite.

2.2.3 MATRIX BOUNDS

The next simple statement gives the key to the problem of matrix bounds derivation for the solution of the CALE.

Lemma 2.1 Suppose that there exist symmetric matrices P_L, P_U such that

$$A^T P_U + P_U A \le -Q \le A^T P_L + P_L A \quad (2.22)$$

Then, $P_L \le P \le P_U$.

Proof. Consider the CALE (1.2), rewritten as follows:

$$A^T (P - P_L) + (P - P_L) A = -(Q + A^T P_L + P_L A) \quad (2.23)$$

$$A^T (P_U - P) + (P_U - P) A = Q + A^T P_U + P_U A \quad (2.24)$$

If the inequalities in (2.22) hold, it follows that $P \ge P_L$ and $P \le P_U$, in accordance with Theorem 1.7.

Remark 2.4 The problem of computing matrix bounds for the CALE solution reduces to the solution of the problem (2.22) for some symmetric matrices P_L, P_U. It is desired to do this without performing some additional computational procedure, e.g. LMI solution (Remark 1.2).

If $Q > 0$, it follows from (2.22) that P_U must be a Lyapunov matrix for A.

Theorem 2.10 Suppose that the symmetric part of A in (1.2) is negative definite. Then,

$$P \le P_{U1}, \quad P_{U1} = «_1 A^T A, \quad «_1 = \frac{1}{2} ª_1[-Q(A^T A_S A)^{-1}] \quad (2.25)$$

$$P \le P_{U2}, \quad P_{U2} = «_2 (AA^T)^{-1}, \quad «_2 = \frac{1}{2} ª_1\{-Q[(AA^T)^{-1} A]_S\} \quad (2.26)$$

Proof. Matrices $A^T A, (AA^T)^{-1}$ are Lyapunov matrices for A if and only if $A \in \boldsymbol{H}^-$, i.e. $A_S < 0$. From the definition of the scalar $«_1$ in (2.25) it follows that

$$«_1 I \ge \frac{1}{2} (-A^T A_S A)^{-1/2} Q (-A^T A_S A)^{-1/2} \Rightarrow -2«_1 A^T A_S A \ge Q \Rightarrow A^T P_{U1} + P_{U1} A + Q \le 0$$

Therefore, P_{U1} is an upper bound for the CALE solution in accordance with Lemma 2.1. The upper matrix bound in (2.26) is proved using similar arguments.

The requirement for $A_S < 0$ is a rather restrictive one. Usage of the singular value decomposition approach helps to get less conservative upper trace and eigenvalue bounds for P.

The next result illustrates the application of this approach for the derivation of upper matrix solution estimates.

Theorem 2.11 Suppose that $A \in \tilde{\boldsymbol{H}}$. Then, the solution matrix in (1.2) can be bounded from above as follows:

$$P \leq P_{U3}, \quad P_{U3} = «_{U3} R_1^{-1}, \quad «_{U3} = \frac{1}{2} ª_1 \{-Q(R_1^{-1}A)^{-1}]_S\}, \quad R_1 = (AA^T)^{1/2} \qquad (2.27)$$

$$P \leq P_{U4}, \quad P_{U4} = «_{U4} R_2, \quad «_{U4} = \frac{1}{2} ª_1 \{-Q[(R_2 A)^{-1}]_S\}, \quad R_2 = (A^T A)^{1/2} \qquad (2.28)$$

Proof. Matrix $A \in \tilde{\boldsymbol{H}}$ if and only if R_1^{-1} and R_2 are Lyapunov matrices for A in accordance with Theorem 2.1, statement (iii). Consider the scalar $«_{U3}$ in (2.27). From its definition it follows that

$$-2«_{U3}(R_1^{-1}A)_S = -«_{U3}(A^T R_1^{-1} + R_1^{-1}A) = -(A^T P_{U3} + P_{U3}A) \geq Q$$

According to Lemma 2.1 P_{U3} in (2.27) is an upper matrix bound for the solution of the CALE.

Having in mind the definition of $«_{U4}$ in (2.28) one gets the matrix inequality

$$-2«_{U4}(R_2 A)_S = -«_{U4}(A^T R_2 + R_2 A) = -(A^T P_{U4} + P_{U4}A) \geq Q$$

Application of Lemma 2.1 shows that P_{U4} in (2.28) is an upper bound for P in (1.2), as well.

Remark 2.5 The upper matrix bounds (2.27) and (2.28) hold under the condition $A \in \tilde{\boldsymbol{H}}$. In view of Theorem 2.1 this means that these bounds are less conservative with respect to validity than the similar bounds (2.25), (2.26) and the matrix bound (1.77), which may be improved in sense of tightness as

$$P \leq P_{U5} = \frac{1}{2} ª_1 (-QA_S^{-1}) I \text{ ; if } A \in \boldsymbol{H}^- \qquad (2.29)$$

It easy to see that

$$a_1(-QA_S^{-1}) \leq a_1(Q) a_1(-A_S^{-1}) = \frac{a_1(Q)}{a_n(-A_S)} = \frac{a_1(Q)}{-a_1(A_S)}$$

Therefore P_{U5} is always tighter than the suggested upper matrix bound (1.77). Both bounds are computable under the one and the same restrictive condition ($A \in \boldsymbol{H}^-$).

The solution bounds (2.27) and (2.28) can be used to derive different upper trace and eigenvalue bounds for the solution of the CALE. They are summarized in the following result.

Lemma 2.2 Let $A \in \tilde{\boldsymbol{H}}$. The following scalar solution bounds hold for the CALE:

$$tr(P) \leq t_{U2} = \min(u_4, u_5) \tag{2.30}$$

where the trace bounds u_4, u_5 are defined as follows

$$u_4 = \frac{1}{2} «_{U3} \sum_{i=1}^{n} \frac{1}{±_i(A)}, \quad u_5 = tr(P_{U4}) = \frac{1}{2} «_{U4} \sum_{i=1}^{n} ±_i(A)$$

$$a_1(P) \leq e_{U2} = \min[a_1(P_{U3}),\ a_1(P_{U4})] \tag{2.31}$$

$$a_1(P_{U3}) = \frac{«_{U3}}{2±_n(A)}\ ,\ a_1(P_{U4}) = \frac{«_{U4}}{2} ±_1(A)$$

and $«_{U3}$ and $«_{U4}$ are given in (2.27) and (2.28), respectively.

Proof. The proof is based on the well known fact that for arbitrary symmetric matrices X and Y one has $X \leq Y \Rightarrow tr(X) \leq tr(Y),\ a_1(X) \leq a_1(Y)$.

The problem of getting lower matrix bounds for the CALE solution is always solvable if the right-hand side matrix Q is strictly positive definite. The next theorem illustrates this fact.

Theorem 2.12 Let $Q > 0$ in (1.2) and X be an arbitrary positive definite matrix. Then,

$$P \geq P_{L1} = «_L X > 0\ ,\ «_{L1} = \frac{1}{-2a_n[(XA)_S Q^{-1}]} \tag{2.32}$$

Proof. From the above definition of $«_L$ and P_L it follows that

$$1 = -2a_n[(XA)_S Q^{-1}]«_L \Rightarrow I \geq -«_L Q^{-1/2}(A^T X + XA)Q^{-1/2} = -Q^{-1/2}(A^T P_L + P_L A)Q^{-1/2}$$

or,

$$Q \geq -A^T P_L - P_L A \Rightarrow Q + A^T P_L + P_L A \geq 0 \Rightarrow P \geq P_L$$

in accordance with Lemma 2.1. Since X is arbitrary, but positive definite matrix, then positive definiteness of the matrix bound P_L depends entirely on the sign of the scalar α_L . Suppose that $\alpha_L \leq 0$, which, due to positive definiteness of Q, is possible if and only if

$$2\lambda_n[(XA)_S] = \lambda_n(A^T X + XA) \geq 0 \Leftrightarrow A^T X + XA \geq 0 \Leftrightarrow x^T(A^T X + XA)x \geq 0, \forall x$$

Let x be an arbitrary eigenvector of the stable matrix A, i.e. $Ax = \lambda x,\ \lambda = -\sigma + j\ \ , \sigma > 0$. Then, $-2\sigma x^T Xx \geq 0 \Leftrightarrow x^T Xx \leq 0$, which contradicts the assumption that X is positive definite. Therefore, $\alpha_L > 0$ and P_L is a positive definite lower matrix bound for P in (1.2).

Remark 2.6 If $Q > 0$, various lower matrix bounds for the CALE solution can be obtained, e.g. see (1.69)-(1.75). Generally speaking, they are based on the usage of a scalar or matrix parameter M and some of the well known matrix inequalities

$$Q = -A^T P - PA \leq PAM^{-1}A^T P + M, \quad 0 < M < Q$$

$$Q = -A^T P - PA \leq A^T M^{-1} A + PMP, \quad M > AQ^{-1}A^T$$

In accordance with the fact that $X \geq Y > 0$ implies $X^{1/2} \geq Y^{1/2}$ (but not *vice versa*) respective bounds for P are obtained. It will be shown later on why the approach applied in Theorem 2.11 to obtain lower matrix bounds is preferable.

A way to overcome the lower matrix bound problem for the CALE when Q is a singular matrix is to use the equivalent DALE-type representation (1.93) of (1.2). This helped to derive the always valid lower bound (1.97). Another possibility lies in the usage of the CALE-type equation (2.20).

Corollary 2.2 Suppose that $\tilde{Q} = Q + A^T QA > 0$. Then the CALE solution has the following matrix and scalar lower bounds:

$$P \geq P_{L2} = \alpha_L X > 0\,;\ \alpha_{L2} = \frac{1}{-2\lambda_n[(X\bar{A})_S \tilde{Q}^{-1}]},\ \bar{A} = A + A^{-1} \tag{2.33}$$

for any positive matrix X, and

$$tr(P) \geq t_{L2} = tr(P_{L2}), \quad \lambda_n(P) \geq e_{L4} = \lambda_n(P_{L2}) \tag{2.34}$$

Proof. It is entirely based on Theorem 2.12 and is omitted.

2.3 IMPROVEMENT OF MATRIX BOUNDS

It will be shown how under the assumption that some lower and upper matrix bounds for the CALE solution are available, they can be additionally improved in sense of tightness in this section.

Theorem 2.13 [125] Let there exist some symmetric matrices P_L, P_U such that $A^T P_U + P_U A \leq -Q \leq A^T P_L + P_L A$ **.** Then, the solution of the CALE can be bounded as:

$$P \geq L_i = (\tilde{A}^i)^T P_L \tilde{A}^i + \sum_{j=0}^{i-1} (\tilde{A}^j)^T \tilde{Q} \tilde{A}^j, \forall i = 0,1,2,.. \tag{2.35}$$

$$P \leq U_i = (\tilde{A}^i)^T P_U \tilde{A}^i + \sum_{j=0}^{i-1} (\tilde{A}^j)^T \tilde{Q} \tilde{A}^j, \forall i = 0,1,2,.. \tag{2.36}$$

where

$$\tilde{A} = (A - I)^{-1}(A + I), \quad \tilde{Q} = 2(A - I)^{-T} Q (A - I)^{-1} \tag{2.37}$$

Further,

$$L_i \geq L_{i-1}, \quad \forall i = 1, 2,..., \quad L_0 = P_L \tag{2.38}$$

$$U_i \leq U_{i-1}, \quad \forall i = 1, 2,..., \quad U_0 = P_U \tag{2.39}$$

Proof. Consider (1.93), where P denotes the solution of the CALE (1.2). For $\sim = 1$ one gets the coefficient and the right hand-side matrices in (2.37). If the supposition of the Theorem holds then P_L, P_U are a lower and an upper matrix bound for the CALE solution, respectively, in accordance with Lemma 2.1.

From the DALE $P = \tilde{A}^T P \tilde{A} + \tilde{Q}$ in (1.93) one gets the following lower bounds

$$P \geq \tilde{A}^T P_L \tilde{A} + \tilde{Q} = L_1 \,\mathbf{;}$$

$$P \geq (\tilde{A}^2)^T P_L \tilde{A}^2 + \tilde{A}^T \tilde{Q} \tilde{A} + \tilde{Q} = L_2 \,\mathbf{;}$$

$$P \geq (\tilde{A}^3)^T P_L \tilde{A}^3 + (\tilde{A}^2)^T \tilde{Q} \tilde{A}^2 + \tilde{A}^T \tilde{Q} \tilde{A} + \tilde{Q} = L_3$$

.....

$$P \geq L_i = (\tilde{A}^i)^T P_L \tilde{A}^i + \sum_{j=0}^{i-1} \tilde{A}^{jT} \tilde{Q} \tilde{A}^j, \forall i = 0,1,2,..$$

and the upper bounds

$$P \leq \tilde{A}^T P_U \tilde{A} + \tilde{Q} = U_1$$

$$P \le (\tilde{A}^2)^T P_U \tilde{A}^2 + \tilde{A}^T \tilde{Q}\tilde{A} + \tilde{Q} = U_2$$

$$P \le (\tilde{A}^3)^T P_U \tilde{A}^3 + (\tilde{A}^2)^T \tilde{Q}\tilde{A}^2 + \tilde{A}^T \tilde{Q}\tilde{A} + \tilde{Q} = U_3$$

.....

$$P \le U_i = (\tilde{A}^i)^T P_U \tilde{A}^i + \sum_{j=0}^{i-1} \tilde{A}^{jT} \tilde{Q}\tilde{A}^j, \ \forall i = 0,1,2,..$$

This proves the first statement of the Theorem. Denote $\Delta L_{i,i-1} = L_i - L_{i-1}$ and for any given i consider the difference matrix

$$\Delta L_{i,i-1} = (\tilde{A}^i)^T P_L \tilde{A}^i + \sum_{j=0}^{i-1} \tilde{A}^{jT} \tilde{Q}\tilde{A}^j - (\tilde{A}^{i-1})^T P_L \tilde{A}^{i-1} - \sum_{j=0}^{i-2} \tilde{A}^{jT} \tilde{Q}\tilde{A}^j$$

$$= (\tilde{A}^i)^T P_L \tilde{A}^i + (\tilde{A}^{i-1})^T \tilde{Q}\tilde{A}^{i-1} - (\tilde{A}^{i-1})^T P_L \tilde{A}^{i-1}$$

$$= (\tilde{A}^{i-1})^T (\tilde{A}^T P_L \tilde{A} + \tilde{Q} - P_L)\tilde{A}^{i-1}$$

Having in mind (2.37), one gets

$$\tilde{A}^T P_L \tilde{A} + \tilde{Q} - P_L = (A-I)^{-T}[(A+I)^T P_L (A+I) + 2Q - (A-I)^T P_L (A-I)](A-I)^{-1}$$

$$= 2(A-I)^{-T}(A^T P_L + P_L A + Q)(A-I)^{-1} \ge 0$$

in accordance with the supposition made in the statement of the Theorem. Therefore,

$$\Delta L_{i,i-1} \ge 0 \Rightarrow L_i \ge L_{i-1}, \quad \forall i = 1,2,..; \quad L_0 \equiv P_L$$

This proves the inequalities in (2.38). Denote $\Delta U_{i,i-1} = U_i - U_{i-1}$. In a similar way one gets

$$\Delta U_{i,i-1} = (\tilde{A}^i)^T P_U \tilde{A}^i + \sum_{j=0}^{i-1} \tilde{A}^{jT} \tilde{Q}\tilde{A}^j - (\tilde{A}^{i-1})^T P_U \tilde{A}^{i-1} - \sum_{j=0}^{i-2} \tilde{A}^{jT} \tilde{Q}\tilde{A}^j$$

$$= (\tilde{A}^i)^T P_U \tilde{A}^i + (\tilde{A}^{i-1})^T \tilde{Q}\tilde{A}^{i-1} - (\tilde{A}^{i-1})^T P_U \tilde{A}^{i-1}$$

$$= (\tilde{A}^{i-1})^T (\tilde{A}^T P_U \tilde{A} + \tilde{Q} - P_U)\tilde{A}^{i-1}$$

Since

$$\tilde{A}^T P_U \tilde{A} + \tilde{Q} - P_U = (A-I)^{-T}[(A+I)^T P_U (A+I) + 2Q - (A-I)^T P_U (A-I)](A-I)^{-1}$$

$$= 2(A-I)^{-T}(A^T P_U + P_U A + Q)(A-I)^{-1} \le 0$$

one finally gets

$$\Delta U_{i,i-1} \le 0 \Rightarrow U_i \le U_{i-1}, \quad \forall i = 1,2,..; \quad U_0 \equiv P_U$$

This completes the proof of the inequalities in (2.39) and the Theorem.

Lemma 2.3 For arbitrary given positive (semi)-definite matrix Q, such that the CALE (1.2) has a positive definite solution, the matrix inequalities in (2.35) are satisfied for $P_L = L_0 = \tilde{Q}$.

Proof. From the proof of Theorem 2.13 it follows that the inequalities (2.35) hold if and only if $Q + A^T P_L + P_L A \geq 0$ **,** where P_L is obviously a lower bound for the CALE solution in accordance with Lemma 2.1. Since $\tilde{Q}$ in (1.93) is a lower matrix bound for P for any positive scalar $\sim$, it remains to prove that $A^T \tilde{Q} + \tilde{Q} A + Q \geq 0$. From the definition of matrix $\tilde{Q}$ in (2.37) one gets

$$A^T \tilde{Q} + \tilde{Q} A + Q = 2[A^T (A - I)^{-T} Q (A - I)^{-1} + (A - I)^{-T} Q (A - I)^{-1} A] + Q = S$$

Note, that matrices $(A - I)^{-1}$ and A commute, i.e. $(A - I)^{-1} A = A(A - I)^{-1}$. Therefore,

$$\begin{aligned} S &= (A - I)^{-T} [2(A^T Q + QA) + (A - I)^T Q (A - I)](A - I)^{-1} \\ &= (A - I)^{-T} (A^T QA + A^T Q + QA + Q)(A - I)^{-1} \\ &= (A - I)^{-T} (A + I)^T Q (A + I)(A - I)^{-1} = \tilde{A}^T Q \tilde{A} \geq 0 \end{aligned}$$

It follows that $A^T \tilde{Q} + \tilde{Q} A + Q \geq 0$ and this proves the statement of the Lemma.

Remark 2.7 Theorem 2.13 states that under some matrix inequality conditions any lower and upper matrix bound can be additionally improved in sense of tightness. These inequalities play an essential role since they guarantee the successive improvement of any given bound satisfying them. Lower matrix bounds can be obtained using different approaches. The estimates (2.32), (2.33) are preferable to the similar ones in (1.69)-(1.75) (see Remark 2.6).

Lemma 2.4 Suppose that Q in (1.2) is a positive semi-definite matrix and (Q, A) is an observable pair. For some integer $i \leq n$ the lower solution bound L_i in (2.35) is a positive definite matrix for any given $P_L = L_0$.

Proof. The following statements are equivalent for $(Q, A), Q = C^T C \geq 0$ to be an observable pair:

(i) $rankM = n, \quad M = [A^T - a^* I \quad C^T] \in \boldsymbol{R}_{n,2n}, \quad \forall a^* \in \pm(A)$

(ii) $rankO = n, \quad O = [C^T \quad A^T C^T \quad (A^2)^T C^T \ \ldots \ (A^{n-1})^T C^T] \in \boldsymbol{R}_{n,nn}$

where $\pm(A)$ denotes the set of eigenvalues of *A*. Consider (i) applied for the pair $(\tilde{A}, \tilde{Q})$ in (2.37), i.e.

$$rank\tilde{M} = n, \quad \tilde{M} = [\tilde{A}^T - a_d^* I \quad \tilde{C}^T] \in \boldsymbol{R}_{n,2n}, \; \tilde{C}^T = \sqrt{2}(A-I)^{-T} Q^{1/2}$$

where a_d denotes eigenvalue of the transformed matrix $\tilde{A}$. Matrix $\tilde{M}$ takes the form

$$\tilde{M} = [(A-I)^{-T}(A^T+I) - a_d^* I \quad \sqrt{2}(A-I)^{-T} Q^{1/2}] = (A-I)^{-T}[A^T + I - a^*(A^T - I) \quad \sqrt{2}Q^{1/2}]$$

$$= (A-I)^{-T}[(1-a_d^*)A^T + (1+a^*)I \quad \sqrt{2}Q^{1/2}]$$

$$= (1-a_d^*)(A-I)^{-T}[A^T + z^* I \qquad {}^* Q^{1/2}], \quad z = \frac{1+a_d}{1-a_d}, \qquad {}^* = \frac{\sqrt{2}}{1-a_d^*}$$

Having in mind that the eigenvalue a_d of $\tilde{A}$ is

$$a_d = \frac{a+1}{a-1}$$

it follows that

$$z = -a$$

or

$$\tilde{M} = (1-a_d^*)(A-I)^{-T}[A^T - a^* I \qquad {}^* Q^{1/2}]$$

Then, $rank\,\tilde{M} = n$ if and only if

$$rank[A^T - a^* I \qquad {}^* Q^{1/2}] = n$$

The above condition is equivalent to observability of the pair $(| \quad |^2 Q, A)$. From statement (ii) it follows that the coefficient   cannot affect the observability of pair (*Q*, *A*). Therefore, the pair (*Q*, *A*) is observable if and only if the pair $(\tilde{Q}, \tilde{A})$ is observable. Then, according to the supposition of the Lemma, the transformed matrix pair is observable. From (1.10), (1.93) and the applied for the transformed pair statement (ii), it follows that the solution of the CALE (1.2) can be bounded from below as follows:

$$P = \sum_{i=0}^{\infty} (\tilde{A}^i)^T \tilde{Q} \tilde{A}^i \geq \sum_{i=0}^{n-1} (\tilde{A}^i)^T \tilde{Q} \tilde{A}^i = \tilde{O}^T \tilde{O} > 0,$$

Consider (2.35) for *i* = *n*, i.e.

$$P \geq L_n = (\tilde{A}^n)^T P_L \tilde{A}^n + \sum_{i=0}^{n-1} \tilde{A}^{iT} \tilde{Q} \tilde{A}^i = (\tilde{A}^n)^T P_L \tilde{A}^n + \tilde{O}^T \tilde{O} > 0$$

Therefore, there exists some positive integer $i \le n$ which guarantees the statement of the Lemma for any given positive (semi)- definite P_L .

Remark 2.8 Up to our best knowledge, the lower matrix bound (2.35) for the CALE solution is the first one which is proven to be strictly positive for some $i \le n$. Also, this bound is applicable for any initial lower matrix bound $P_L = L_0 \ge 0$ for P. If $P_L = L_0 = \tilde{Q} \ge 0$ then

$$P \ge L_{n-1} = (\tilde{A}^{n-1})^T \tilde{Q}\tilde{A}^n + \sum_{i=0}^{n-2} \tilde{A}^{iT} \tilde{Q}\tilde{A}^i = \tilde{O}^T \tilde{O} > 0 \qquad (2.40)$$

and therefore the existence of a strictly positive lower matrix bound for the CALE solution is guaranteed for some $i \le n-1$.

Remark 2.9 Consider the lower matrix bounds for the CALE solution (2.35). Under the supposition that Q is a positive (semi)-definite matrix and (Q, A) is an observable pair, there exists some $i \le n$, such that

$$\text{a}_n(P) \ge e_{L5,i} = \text{a}_n(L_i) = \text{a}_n[(\tilde{A}^i)^T P_L \tilde{A}^i + \sum_{j=0}^{i-1} \tilde{A}^{jT} \tilde{Q}\tilde{A}^j] > 0, \ \forall i = 0,1,2.. \qquad (2.41)$$

for any given matrix positive (semi)-definite matrix P_L . If in addition, the inequality $Q + A^T P_L + P_L A \ge 0$ holds, then

$$e_{L5,i} \ge e_{L5,i-1}, \forall i$$

$$tr(P) \ge t_{L3,i} = tr(L_i) \ge tr(L_{i-1}), \forall i$$

Having in mind (2.36), the following upper eigenvalue and trace bounds for the CALE solution hold:

$$\text{a}_1(P) \le e_{U3,i} = \text{a}_1[(\tilde{A}^i)^T P_U \tilde{A}^i + \sum_{j=0}^{i-1} \tilde{A}^{jT} \tilde{Q}\tilde{A}^j], \forall i = 0,1,2.. \qquad (2.42)$$

$$tr(P) \le t_{U3,i} = tr[(\tilde{A}^i)^T P_U \tilde{A}^i + \sum_{j=0}^{i-1} \tilde{A}^{jT} \tilde{Q}\tilde{A}^j], \forall i = 0,1,2,.. \qquad (2.43)$$

$$e_{U3,i} \le e_{U3,i-1}, \quad t_{U3,i} \le t_{U3,i-1}, \quad \forall i$$

for arbitrary P_U satisfying the matrix inequality $A^T P_U + P_U A + Q \le 0$.

The main difficulty arising in estimating the CALE solution from below is due to the fact that the guaranteeing positive definiteness of P observability condition is usually not taken into account. Exceptions in this sense are the bounds due to Corollary 2.1 and Lemma 2.4. The next result is another attempt to overcome the problem with the singularity of matrix Q.

Theorem 2.14 Suppose that (Q, A) is an observable pair. If some n x n matrix K exists such that $A_C = -(A + KQ) \in \tilde{H}$, then the solution of the CALE (1.2) has the following bounds:

$$P \geq P_{L3} = \frac{1}{«_1}\tilde{R}_1^{-1}, \quad \tilde{R}_1 = (A_C A_C^T)^{1/2}, \quad «_1 = 2ª_1[-KQK^T(A_C\tilde{R}_1)^{-1}] \qquad (2.44)$$

$$P \geq P_{L4} = \frac{1}{«_2}\tilde{R}_2, \quad \tilde{R}_2 = (A_C^T A_C)^{1/2}, \quad «_2 = 2ª_1[-KQK^T(A_C\tilde{R}_2^{-1})^{-1}] \qquad (2.45)$$

Proof. If (Q, A) is an observable pair the solution of the CALE is a positive definite matrix. Then, (1.2) can be rewritten as:

$$P^{-1}(-A - KQ)^T + (-A - KQ)P^{-1} = P^{-1}A_C^T + A_C P^{-1} = (P^{-1} - K)Q(P^{-1} - K^T) - KQK^T = -\tilde{Q}$$

The conditions for observability of (Q, A) is equivalent to the condition for controllability of the pair (A^T, Q) , or in other words there always exists some matrix K such that $A_C = -(A + KQ)$ is a stable matrix. Consider the singular value decomposition of matrix A_C i.e.,

$$A = \tilde{R}_1\tilde{F} = \tilde{F}\tilde{R}_2, \ \tilde{R}_1 = (A_C A_C^T)^{1/2}, \quad \tilde{R}_2 = (A_C^T A_C)^{1/2}, \quad \tilde{F}^T\tilde{F} = I$$

and suppose that $A_C \in \tilde{H}$. According to Theorem 2.1, $\tilde{R}_1^{-1}$ and $\tilde{R}_2$ are Lyapunov matrices for A_C , or equivalently, $\tilde{R}_1$ and $\tilde{R}_2^{-1}$ are Lyapunov matrices for A_C^T (Definition 1.3). This, having in mind the defined in (2.44) and (2.45) scalars $«_1$, $«_2$, guarantees the inequalities

$$0 \geq «_1(\tilde{R}_1 A_C^T + A_C\tilde{R}_1) + KQK^T = \tilde{P}_1 A_C^T + A_C\tilde{P}_1 + KQK^T \geq \tilde{P}_1 A_C^T + A_C\tilde{P}_1 + \tilde{Q}$$

$$0 \geq «_2(\tilde{R}_2^{-1} A_C^T + A_C\tilde{R}_2^{-1}) + KQK^T = \tilde{P}_2 A_C^T + A_C\tilde{P}_2 + KQK^T \geq \tilde{P}_2 A_C^T + A_C\tilde{P}_2 + \tilde{Q}$$

where $\tilde{P}_1 = «_1\tilde{R}_1$, $\tilde{P}_2 = «_2\tilde{R}_2^{-1}$. From Lemma 2.1 it follows that $\tilde{P}_1$, $\tilde{P}_2$ are both upper bounds for P^{-1} , or their inverses, denoted P_{L3} and P_{L4} , respectively, are lower bounds for the solution of the CALE (1.2). This completes the proof of the Theorem.

Remark 2.10 The validity of the lower matrix bounds (2.44) and (2.45) depends only on the condition $A_C = -(A + KQ) \in \tilde{H}$, which is less conservative than the usual requirement

$A_C = -(A + KQ) \in H^-$. If this condition is satisfied, than two positive definite lower matrix bounds for the CALE solution are computable. This makes possible to derive nontrivial lower bounds for the minimal eigenvalue of *P*, as well.

Corollary 2.3 If the suppositions of Theorem 2.14 hold, then the minimal eigenvalue of the positive definite solution of the CALE (1.2) has the lower positive bound

$$\lambda_n(P) \geq \max[\lambda_n(P_{L3}),\ \lambda_n(P_{L4})] \tag{2.46}$$

where matrices P_{L3} and P_{L4} are given in (2.44) and (2.45), respectively.

2.4 IMPROVEMENT OF SCALAR BOUNDS

The problem of trace and eigenvalue bounds improvement for the CALE solution in sense of tightness is discussed. Before that we need the following useful result due to [77]. Let *X* and *Y*, $Y \geq 0$, be some *n* x *n* symmetric matrices. Then, the trace of the product *XY* can be bounded as follows:

$$\sum_{i=1}^{n} \lambda_{n-i+1}(X)\lambda_i(Y) \leq tr(XY) \leq \sum_{i=1}^{n} \lambda_i(X)\lambda_i(Y) \tag{2.47}$$

For arbitrary symmetric matrices X and $Y \geq 0$ define the nonnegative scalars

$$\delta_1(X,Y) = \sum_{i=2}^{n} [\lambda_1(X) - \lambda_i(X)]l_i\,,\ \ \delta_2(X,Y) = \sum_{i=2}^{n} [\lambda_{n-i+1}(X) - \lambda_n(X)]l_i \tag{2.48}$$

where $\lambda_i(Y) \geq l_i \geq 0,\ i = 2,..,n$.

Some of the results given below are new, and the others were presented firstly in [117], [120], [123].

Theorem 2.15 Let l_i, denotes some nonnegative lower bound for the *i*-th eigenvalue of the solution *P* in (1.2), $i = 2,..,n$. Then, having in mind (2.48), the trace of the CARE solution can be bounded as follows:

$$\frac{\frac{1}{2}tr(Q) + \delta_2(A_S, P)}{-\lambda_n(A_S)} = t_{L4} \leq tr(P) \leq t_{U4}\ \frac{\frac{1}{2}tr(Q) - \delta_1(A_S, P)}{-\lambda_1(A_S)} \tag{2.49}$$

$$\frac{\frac{1}{2}tr(R_1Q) + \delta_2(R_1F_SR_1, P)}{-\lambda_n(R_1F_SR_1)} \leq t_{L5} = tr(P) \leq t_{U5} = \frac{\frac{1}{2}tr(R_1Q) - \delta_1(R_1F_SR_1, P)}{-\lambda_1(R_1F_SR_1)} \tag{2.50}$$

$$\frac{\frac{1}{2}tr(R_2^{-1}Q)+\text{¥}_2(F_S,P)}{-\text{ª}_n(F_S)} \le t_{L6} = tr(P) \le t_{U6}\,\frac{\frac{1}{2}tr(R_2^{-1}Q)-\text{¥}_1(F_S,P)}{-\text{ª}_1(F_S)} \tag{2.51}$$

The upper estimate in (2.49) holds if $A \in \boldsymbol{H}^-$ and $A \in \tilde{\boldsymbol{H}}$ is the condition under which the upper estimates in (2.50) and (2.51) are valid.

Proof. Pre-multiplication of the CALEs (2.5) by R_1 and (2.7) by R_2^{-1}, respectively, leads to the matrix equalities:

$$R_1F^TR_1P + R_1PR_1F = -R_1Q \tag{2.52}$$

$$F^TP + R_2^{-1}PFR_2 = -R_2^{-1}Q \tag{2.53}$$

Application of the trace operator to both sides of (1.2), (2.52) and (2.53) results in

$$2tr(-A_SP) = tr(Q)$$

$$2tr(-R_1F_SR_1P) = tr(R_1Q)$$

$$2tr(-F_SP) = tr(R_2^{-1}Q)$$

Using (2.47), the following trace inequalities are obtained:

$$\frac{1}{2}tr(Q) = tr(-A_SP) \ge \sum_{i=1}^{n}\text{ª}_{n-i+1}(-A_S)\text{ª}_i(P) = -\sum_{i=1}^{n}\text{ª}_i(A_S)\text{ª}_i(P)$$

$$= -\text{ª}_1(A_S)tr(P) + \sum_{i=2}^{n}[\text{ª}_1(A_S) - \text{ª}_i(A_S)]\text{ª}_i(P)$$

$$\ge -\text{ª}_1(A_S)tr(P) + \sum_{i=2}^{n}[\text{ª}_1(A_S) - \text{ª}_i(A_S)]l_i$$

$$= -\text{ª}_1(A_S)tr(P) + \text{¥}_1(A_S,P) \tag{2.54}$$

$$\frac{1}{2}tr(Q) = tr(-A_SP) \le \sum_{i=1}^{n}\text{ª}_i(-A_S)\text{ª}_i(P)$$

$$= \text{ª}_1(-A_S)tr(P) + \sum_{i=2}^{n}[\text{ª}_n(-A_S) - \text{ª}_1(-A_S)]\text{ª}_i(P)$$

$$= -\text{ª}_n(A_S)tr(P) + \sum_{i=2}^{n}[\text{ª}_n(A_S) - \text{ª}_{n-i+1}(A_S)]\text{ª}_i(P)$$

$$= -\text{ª}_n(A_S)tr(P) - \sum_{i=2}^{n}[\text{ª}_{n-i+1}(A_S) - \text{ª}_n(A_S)]\text{ª}_i(P)$$

$$\leq -\lambda_n(A_S)tr(P) - \sum_{i=2}^{n}[\lambda_{n-i+1}(A_S) - \lambda_n(A_S)]l_i$$

$$= -\lambda_n(A_S)tr(P) - \Delta_2(A_S, P) \qquad (2.55)$$

The bounds in (2.49) are easily obtained from (2.54) and (2.55), respectively. Following the same proof scheme and having in mind (2.48), bounds (2.50) and (2.51) follow easily.

Remark 2.11 The lower and upper trace bounds in (2.49)-(2.51) are always tighter than the respective similar lower estimates (1.20), (1.21), and the upper estimates (1.24), (2.4) and (2.6). This is due to the fact that for any symmetric matrix X the quantities $\Delta_1(X,Y), \Delta_2(X,Y)$ in (2.48) are always nonnegative. If, in addition, all required lower solution bounds l_i are positive, then $\Delta_1(X,Y) = 0 \Leftrightarrow \Delta_2(X,Y) = 0$. If $\Delta_1(X,Y) = 0$, then $X = \sim I$. The respective lower and upper bounds coincide and are equal the solution trace, in this special case. Thus, the usage of any available nonnegative lower bounds for the eigenvalues of the CALE solution matrix helps to improve both lower and upper existing trace estimates.

Available lower eigenvalue bounds are used in Theorem 2.15 to make previous bounds sharper. A lower and an upper matrix bounds are used to derive the next result for the same purpose.

Theorem 2.16 Let

$$Q_L = A^T P_L + P_L A + Q \geq 0$$

$$Q_U = A^T P_U + P_U A + Q \leq 0$$

for some symmetric matrices P_L and P_U. The solution trace can be estimated as follows:

$$tr(P) \geq t_{L7} = tr(P_L) + \frac{1}{-2\lambda_n(A_S)}\varphi(\alpha_L) \qquad (2.56)$$

$$\varphi(\alpha_L) = \frac{tr(AA^T Q_L)\alpha_L^2 + 2tr\langle[A_S - \lambda_n(A_S)I]Q_L\rangle\alpha_L + tr(Q_L)}{\sigma_1^2(A)\alpha_L^2 + 1} \qquad (2.57)$$

$$tr(P) \leq t_{U7} = tr(P_U) + \frac{1}{-2\lambda_n(A_S)}\varphi(\alpha_U) \qquad (2.58)$$

$$\varphi(\alpha_U) = \frac{tr(AA^T Q_U)\alpha_U^2 + 2tr\langle[A_S - \lambda_n(A_S)I]Q_U\rangle\alpha_U + tr(Q_U)}{\sigma_1^2(A)\alpha_U^2 + 1} \qquad (2.59)$$

where the parameters $«_L, «_U$ in (2.57) and (2.59) are defined as:

$$«_L = \frac{- \quad + \sqrt{\ ^2 + \frac{\sim^2}{\pm_1^2(A)}}}{2\sim}; \quad \sim = \pm_1^2(A) tr(\tilde{A}_S Q_L), \quad = tr(\bar{A} Q_L) \qquad (2.60)$$

$$«_U = \frac{- \quad + \sqrt{\ ^2 + \frac{\sim^2}{\pm_1^2(A)}}}{2\sim}; \quad \sim = \pm_1^2(A) tr(-\tilde{A}_S Q_U), \quad = tr(-\bar{A}) Q_U \qquad (2.61)$$

and $\tilde{A}_S = A_S - ª_n(A_S)I$, $\bar{A} = \pm_1^2(A)I - AA^T$.

Proof. It follows that $P \geq P_L$ and $P \leq P_U$ in accordance with the suppositions and Lemma 2.1. Then, some positive (semi)-definite matrices L and U exist, such that $P = P_L + L$ and $P = P_U - U$. The CALE (1.2) can be rewritten as:

$$A^T L + LA = -(Q + A^T P_L + P_L A) = -Q_L \leq 0 \qquad (2.62)$$

$$A^T U + UA = Q + A^T P_U + P_U A = Q_U \leq 0 \qquad (2.63)$$

Therefore, matrices L and U are unique solutions of (2.62) and (2.63), respectively. Consider the following based on them matrix inequalities:

$$«_L^2 A^T LA + L \geq «_L Q_L, \ \forall «_L > 0$$

$$«_U^2 A^T UA + U \geq -«_U Q_U, \ \forall «_U > 0$$

There must exist some positive (semi)-definite matrices $\Delta_L(«_L), \Delta_U(«_U)$ denoted Δ_L, Δ_U for simplicity, such that

$$«_L^2 A^T LA + L = «_L Q_L + \Delta_L \Rightarrow L = «_L Q_L + \Delta_L - «_L^2 A^T LA \qquad (2.64)$$

$$«_U^2 A^T UA + U = -«_U Q_U + \Delta_U \Rightarrow U = -«_U Q_U + \Delta_U - «_U^2 A^T UA \qquad (2.65)$$

Substitution of L and M in (2.62) and (2.63) leads to a new couple of CALEs

$$A^T \Delta_L + \Delta_L A = -Q_L - «_L(A^T Q_L + Q_L A) + «_L^2 A^T (A^T L + LA) A$$

$$A^T \Delta_U + \Delta_U A = Q_U + «_U(A^T Q_U + Q_U A) + «_U^2 A^T (A^T U + UA) A$$

In view of (2.62) and (2.63) the above CALEs take the form:

$$A^T \Delta_L + \Delta_L A = -(«_L A^T + I) Q_L («_L A + I) = -\tilde{Q}_L \leq 0 \qquad (2.66)$$

$$A^T \Delta_U + \Delta_U A = («_U A^T + I) Q_U («_U A + I) = \tilde{Q}_U \leq 0 \qquad (2.67)$$

By making use of (1.20) one gets the following nonnegative lower bounds for the traces of Δ_L and Δ_U

$$tr(\Delta_L) \geq \frac{tr(\tilde{Q}_L)}{-2ª_n(A_S)}; \qquad tr(\Delta_U) \geq \frac{-tr(\tilde{Q}_U)}{-2ª_n(A_S)} \tag{2.68}$$

Taking into account (2.64) and (2.65) one finally gets the following expressions for the traces of L and U:

$$tr(«_L^2 A^T LA + L) = tr[(«_L^2 \; AA^T + I)L] = «_L tr(Q_L) + tr(\Delta_L)$$

$$tr(«_U^2 A^T UA + U) = tr[(«_U^2 \; AA^T + I)L] = -«_U tr(Q_U) + tr(\Delta_U)$$

which result in respective trace inequalitites

$$tr(L) \geq \frac{«_L tr(Q_L) + tr(\Delta_L)}{1 + ±_1^2(A)«_L^2}$$

$$tr(U) \geq \frac{-«_U tr(Q_U) + tr(\Delta_U)}{1 + ±_1^2(A)«_U^2}$$

Since $P = P_L + L$ and $P = P_U - U$, the next lower and upper bounds are obtained

$$tr(P) = tr(P_L) + tr(L) \geq tr(P_L) + \frac{«_L tr(Q_L) + tr(\Delta_L)}{1 + ±_1^2(A)«_L^2} \tag{2.69}$$

$$tr(P) = tr(P_U) - tr(U) \leq tr(P_U) + \frac{«_U tr(Q_U) - tr(\Delta_U)}{1 + ±_1^2(A)«_U^2} \tag{2.70}$$

With the estimates (2.68) for the scalars $tr(\Delta_L)$ and $tr(\Delta_U)$ the trace inequalities (2.69) and (2.70) can be rewritten as follows:

$$tr(P) \geq tr(P_L) + \frac{«_L tr(Q_L) + \dfrac{tr(\tilde{Q}_L)}{-2ª_n(A_S)}}{1 + ±_1^2(A)«_L^2} = tr(P_L) + \frac{1}{-2ª_n(A_S)} ¨(«_L)$$

where $¨(«_L)$ is given in (2.57) and

$$tr(P) \leq tr(P_U) + \frac{«_U tr(Q_U) + \dfrac{tr(\tilde{Q}_U)}{-2ª_n(A_S)}}{1 + ±_1^2(A)«_U^2} = tr(P_U) + \frac{1}{-2ª_n(A_S)} ¨(«_U)$$

and $¨(«_U)$ is defined in (2.58).

The rational functions ¨ $(«_L)$ and $-$¨ $(«_U)$ are always nonnegative since $Q_L \geq 0$ and $Q_U \leq 0$ by assumption and $X -$ ª$_n(X)I \geq 0$ for any given symmetric matrix *X*. Simple computations show that that their maximal values are achieved for $«_L$ and $«_U$ defined in (2.60) and (2.61) respectively. This proves the statements of the Theorem.

Remark 2.12 The lower and upper trace bounds in (2.56) and (2.58) are always tighter than the bounds $tr(P_L) \leq tr(P) \leq tr(P_U)$ for any lower and upper matrix solution bounds, satisfying the inequalities in (2.22).

Provided that certain matrix inequalities hold the next result helps to improve lower and upper eigenvalue bounds for the CALE solution.

Lemma 2.5 Let P_L and P_U denote some lower and matrix bound, respectively, for the solution of the CALE (1.2). The following eigenvalue bounds hold for *P*

$$ª_n(P) \geq e_{L6} = \frac{1}{2} ª_n[(Q + 2\sim P_L)(\sim I - A_S)^{-1}] \qquad (2.71)$$

$$ª_1(P) = e_{U4} \leq \frac{1}{2} ª_1[(Q + 2\ \ P_U)(\ \ I - A_S)^{-1}] \qquad (2.72)$$

for any positive parameters $\sim\ > ª_1(A_S)$ and $\ \ > ª_1(A_S)$. If, in addition there exist positive scalars $\sim^*$ and $\ ^*$ such that

$$Q + 2\sim^*[P_L - ª_n(P_L)I] + 2ª_n(P_L)A_S \geq 0 \qquad (2.73)$$

$$Q + 2\ \ ^*[P_U - ª_1(P_U)I] + 2ª_1(P_U)A_S \leq 0 \qquad (2.74)$$

the bounds in (2.71) and (2.72) satisfy the inequalities:

$$e_{L6} \geq ª_n(P_L), \quad \sim\ \geq \max[ª_1(A_S), \sim^*] \qquad (2.75)$$

$$e_{U4} \leq ª_1(P_U), \quad \ \ \geq \max[ª_1(A_S), \ \ ^*] \qquad (2.76)$$

Proof. For any given scalar parameter® the CALE (1.2) can be rewritten as:

$$(®I - A)^T P + P(®I - A) = Q + 2®P$$

Let ® be chosen to satisfy the inequality$® > ª_1(A_S)$, which guarantees that $®I - A_S > 0$

If *x* is an eigenvector for *P* any eigenvalue of the solution matrix satisfies the equality

$$2\lambda(P)=\frac{y^T(\alpha I-A_S)^{-1/2}(Q+2\alpha P)(\alpha I-A_S)^{-1/2}y}{y^Ty};\;\; Px=\lambda(P)x,\;\; (\alpha I-A_S)^{1/2}x=y$$

and can be estimated as follows:

$$\lambda_n[(Q+2\alpha P)(\alpha I-A_S)^{-1}]\le 2\lambda(P)\le \lambda_1[(Q+2\alpha P)(\alpha I-A_S)^{-1}]$$

If P_L and P_U are such that $P_L\le P\le P_U$, then for some positive scalars β and γ satisfying $\beta>\lambda_1(A_S)$ and $\gamma>\lambda_1(A_S)$ any solution eigenvalue satisfies the inequalities

$$\lambda_n[(Q+2\beta P_L)(\beta I-A_S)^{-1}]\le 2\lambda(P)\le \lambda_1[(Q+2\gamma P_U)(\gamma I-A_S)^{-1}]$$

which proves the bounds in (2.71) and (2.72). The inequalities in (2.75) and (2.76) hold if and only if (2.73) and (2.74), respectively, are satisfied. This proves the statements of the Lemma.

2.5 UNCONDITIONAL INTERNAL UPPER BOUNDS

All known upper bounds for the CALE depend on some, more or less, conservative condition concerning the coefficient matrix *A*. Motivated by this fact an approach to derive always computable upper eigenvalue, trace and matrix solution bounds for *P* in (1.2) is suggested. These bounds use only the fact that *A* is a stable matrix, which guarantees that a unique positive (semi)-definite solution of the CALE exists.

Consider the Schur decomposition of the stable matrix A [49], i.e. $A=U^*TU$, where *U* is a unitary matrix and $T=[t_{ij}]$ is an upper triangular matrix defined as follows:

$$T=[t_{ij}],\quad t_{ij}=\begin{cases}\lambda_i(A), & \text{if } i=j\\ t_{ij}, & \text{if } i<j\\ 0, & \text{if } i>j\end{cases};\;\; \lambda_i(A)=-r_i+j\mu_i,\, r_i>0,\, i=1,..,n\Rightarrow T=\Lambda+\tilde{T} \qquad (2.77)$$

where Λ is a diagonal matrix containing the eigenvalues of *A*, and $\tilde{T}$ is an upper triangular matrix, containing the off-diagonal entries of *T*. If *T* is real, then *U* can be chosen as an orthogonal matrix. The following result is essential for the main result.

Lemma 2.6 [126] A *n* x *n* matrix *A* is stable if and only if a positive scalar $\varepsilon\le 1$ exists, such that $\Xi=U^*E^{-2}U$ is a Lyapunov matrix for *A*, where *E* is a diagonal matrix with entries $e_{ii}=\varepsilon^i, i=1,..,n$.

Proof. Let A be stable and consider the matrix

$$A^T\Xi + \Xi A = U^*T^*E^{-2}U + U^*E^{-2}TU = U^*E^{-1}(ET^*E^{-1} + E^{-1}TE)E^{-1}U = U^*E^{-1}\Phi(£)E^{-1}U$$

In view of (2.77) matrix $\Phi(£)$ is defined as follows:

$$\Phi(£) = [¤_{ij}], \quad ¤_{ij} = ¤_{ji} = \begin{cases} -2r_i, & i = j \\ £^{\,j-i}t_{ij}, & i < j \end{cases} \qquad (2.78)$$

This helps representing (2.78) in the form

$$\Phi(£) = -2\Lambda + \sum_{k=1}^{n-1} £^{\,k} T_k \qquad (2.79)$$

where Λ is a diagonal matrix with entries $r_i > 0, i = 1,..,n$, and the entries of the symmetric matrix $T_k, k = 1,..,n-1$, are $t_{ij}, i < j, j - i = k$. Then $\Phi(£)$ is a negative definite matrix if and only if

$$\tilde{\Phi}(£) = -2I + \sum_{k=1}^{n-1} £^{\,k}\tilde{T}_k < 0, \tilde{T}_k = \Lambda^{-1/2}T_k\Lambda^{-1/2} \qquad (2.80)$$

or equivalently,

$$2 > ª_1(\sum_{k=1}^{n-1} £^{\,k}\tilde{T}_k) = ª_1[\Theta(£)], \quad \Theta(£) = \sum_{i=1}^{n-1} £^{\,k}T_k\Lambda^{-1} \qquad (2.81)$$

where

$$\Theta(£) = [⎡_{ij}] = \begin{cases} 0; & i = j \\ \dfrac{£^{\,j-i}t_{ij}}{r_j}; & i < j \\ \dfrac{£^{\,i-j}t_{ji}}{r_i}; & i > j \end{cases} ; \qquad t_{ij} = t_{ji} \qquad (2.82)$$

It is clear that there always exists some sufficiently small positive $£$ satisfying (2.80),or equivalently, (2.81) which proves the necessity part.

Let $\Xi = U^*E^{-2}U$ be a Lyapunov matrix for *A*, i.e., $f(x) = x^T(A^T\Xi + \Xi A)x < 0, \forall x \neq 0$. If x_i is an eigenvector corresponding to $ª_i(A)$, then

$$f(x) = -2r_i x_i^T \Xi x_i < 0 \Leftrightarrow r_i > 0, i = 1,..,n \Rightarrow A \text{ is a stable matrix}$$

This proves the sufficiency part and the statement of the Theorem.

There exist different ways to obtain $\pounds$ satisfying the equivalent matrix inequalities in (2.79) and (2.80). Some of them are given in the next Lemma. Before that we need some preliminary results.

Theorem 2.17 [49] If X is a n x n Hermitian matrix with negative diagonal entries and

$$\left|x_{ii}\right| > \sum_{j=1,\, j\neq i}^{n} \left|x_{ij}\right|,\ i = 1,..,n \qquad (2.83)$$

then X is negative definite. It is said that X is a strongly diagonal dominant matrix.

Theorem 2.18 [49] Denote $\left|X\right| = [\left|x_{ij}\right|]$ and let A and B are arbitrary n x n complex matrices. If $\left|A\right|[\leq]B$, then $°(A) \leq °(\left|A\right|) \leq °(B)$, where $°(X) = \max\left|ª(X)\right|$ is the spectral radius of matrix X and $\left|A\right|[\leq]B$ means that $b_{ij} \geq \left|a_{ij}\right|, \forall i, j$.

Lemma 2.7 $\Xi = U^{*}E^{-2}U$ is a Lyapunov matrix for A if:

(i) $2 > \sum_{k=1}^{n-1} \pounds^{k} ª_{1}(\tilde{T}_{k})$, where matrices $\tilde{T}_{k}$ are defined in (2.79)

(ii) $\pounds < \dfrac{2}{\sum_{k=1}^{n-1} ª_{1}(T_{k})}$

(iii) $\Phi(\pounds)$ in (2.78) is a strongly diagonal dominant matrix, i.e.,

$$2r_{i} > \sum_{j=1}^{n} \pounds^{j-i}\left|t_{ij}\right|, \quad i < j,\ \ 2r_{i} > \sum_{j=1}^{n} \pounds^{i-j}\left|t_{ij}\right|, \quad i > j; \quad i = 1,..,n$$

(iv) $\pounds < \min \dfrac{2r_{i}}{\sum_{j=1,\, j\neq i}^{n}\left|t_{ij}\right|}$

(v) $\pounds < \dfrac{1}{°[\left|\tilde{T}_{S}\right|\Lambda^{-1}]}$, where $\tilde{T}_{S}$ is the symmetric part of $\tilde{T}$ obtained from T by setting its diagonal entries $ª_{i}(A) = 0, i = 1,..,n$.

(vi) $\left\|\pounds_{v}\right\| < \dfrac{2}{\pm_{1}(\tilde{T})}$; $\quad \pounds_{v} = (\pounds\ \pounds^{2} ... \pounds^{n-1})^{T}, \quad \tilde{T} = [\tilde{T}_{1}\ \tilde{T}_{2} ... \tilde{T}_{n-1}] \in \boldsymbol{R}_{n,n(n-1)}$

and matrices $\tilde{T}_{k}, k = 1,..,n-1$ are defined in (2.79).

Proof.

(i) Taking into account the well known inequality $\mathrm{a}_1(\sum_{i=1}^{N} X_i) \leq \sum_{i=1}^{N} \mathrm{a}_1(X_i)$ it follows directly from (2.80).

(ii) If $A \in \boldsymbol{H}^-$, then for $\pounds = 1$, matrix $\Xi = U^* E^{-2} U = I$ is a Lyapunov matrix for *A*. In the opposite case, a solution exists if $\pounds < 1$. Therefore,

$$\sum_{k=1}^{n-1} \pounds^k \mathrm{a}_1(\tilde{T}_k) \leq \pounds \sum_{k=1}^{n-1} \mathrm{a}_1(\tilde{T}_k)$$

since $\mathrm{a}_1(\tilde{T}_k) \geq 0,\ k = 1,..,n-1$, and the statement is proved.

(iii) It follows from Theorem 2.16.

(iv) Using similar arguments as in the proof of (ii) the statement follows.

(v) Consider the inequality condition (2.81). From the definition of the spectral radius and Theorem 2.18 one has

$$\mathrm{a}_1(\sum_{k=1}^{n-1} \pounds^k \tilde{T}_k) = \mathrm{a}_1[\Theta(\pounds)] \leq {}^\circ[\Theta(\pounds)] \leq {}^\circ[|\Theta(\pounds)|]$$

In view of (2.82) one gets

$$|\Theta(\pounds)| = [|\ulcorner_{ij}|], \quad |\ulcorner_{ij}| = \begin{cases} 0; & i = j \\ \dfrac{\pounds^{j-i}|t_{ij}|}{r_j}; & i < j \\ \dfrac{\pounds^{i-j}|t_{ji}|}{r_i}; & i > j \end{cases}$$

Let $\tilde{t}_{ij}$ are the entries of $\tilde{T}_S$ .Since $\pounds < 1$ one gets

$$|\ulcorner_{ij}| \leq \tilde{t}_{ij} = \frac{|t_{ij}|}{r_j}, \forall i \neq j \Rightarrow |\Theta(\pounds)|[\leq]\{|\tilde{T}_S|\Lambda^{-1}\},$$

i.e.

$$\mathrm{a}_1(\sum_{k=1}^{n-1} \pounds^k \tilde{T}_k) = \mathrm{a}_1[\Theta(\pounds)] \leq {}^\circ[|\Theta(\pounds)|] \leq \pounds {}^\circ[|\tilde{T}_S|\Lambda^{-1}]$$

in accordance with Theorem 2.18. Therefore, for £ chosen as in (v), the inequality condition (2.81) is sufficiently satisfied.

(vi) The inequality (2.80) holds if and only if

$$2 > ª_1(\sum_{k=1}^{n-1} £^k \tilde{T}_k) = ª_1(\tilde{T}\tilde{E}), \quad \tilde{E} = [£I\ £^2 I..£^{n-1}I]^T \in \boldsymbol{R}_{n(n-1),n}$$

Since $ª_1(X) \le ±_1(X)$ for any symmetric matrix X and $±_1(XY) \le ±_1(X)±_1(X)$ for arbitrary matrices X and Y [94], the above inequality holds by sufficiency if $2 > ±_1(\tilde{T})±_1(\tilde{E}) = ±_1(\tilde{T})\|£_v\| \ge ±_1(\tilde{T}\tilde{E})$, which proves the statement.

Provided that A is a stable matrix a Lyapunov function of the form $\Xi = U^* E^{-2} U$ always exists for it in accordance with Lemma 2.6. Lemma 2.7 suggests several ways to compute it. Now, let the problem be stated in a slightly different way. It is desired to determine a positive scalar $£ < 1$ such that the matrix inequality

$$ET^* E^{-1} + E^{-1}TE + 2D \le 0 \qquad (2.84)$$

holds for some chosen positive diagonal matrix $D = [d_i], d_i < r_i, i = 1,..,n$. Then the matrix in (2.79) takes the similar form

$$\tilde{\Phi}(£) = -2\tilde{\Lambda} + \sum_{k=1}^{n-1} £^k T_k, \quad \tilde{\Lambda} = [\tilde{r}_i] = [r_i - d_i], \tilde{r}_i > 0 \qquad (2.85)$$

The problem of computing some appropriate $£ < 1$ such that $\tilde{\Phi}(£)$ in (2.85) is a negative semi-definite matrix reduces to the application of some of the sufficient conditions stated in Lemma 2.7.

Lemma 2.8 Let the inequality (2.84) holds for some £ . The solution matrix P for the CALE has the following always valid upper bounds:

$$P \le P_{U6} = «_U UE^{-2}U^*, \quad «_{U6} = \frac{1}{2} ª_1(QU^* ED^{-1}EU) \qquad (2.86)$$

$$ª_1(P) \le e_{U5} = ª_1(P_{U6}) = \frac{«_{U6}}{£^{2n}} \qquad (2.87)$$

$$tr(P) \le t_{U8} = tr(P_{U6}) = «_{U6} \sum_{i=1}^{n} \frac{1}{£^{2i}} \qquad (2.88)$$

Proof. Let (2.84) be satisfied, i.e.

$$\tilde{A}^*(«_{U6} I) + («_{U6} I)\tilde{A} \le -2«_U D < 0, \quad \forall «_{U6} > 0, \quad \tilde{A} = E^{-1}TE$$

Having in mind the Schur decomposition $A = U^{*}TU$ of matrix *A*, consider the CALE (1.2) pre- and post-multiplied by matrices $S = EU$ and S^{*}, respectively:

$$\tilde{A}^{*}\tilde{P} + \tilde{P}\tilde{A} = -SQS^{*} = -\tilde{Q}, \quad \tilde{P} = SPS^{*}, \quad \tilde{Q} = SQS^{*}, \quad \tilde{A}_{S} \le D < 0$$

If $«_{U6}$ is chosen as in (2.86), negative semi-definiteness of matrix $-2«_{U6}D + \tilde{Q}$ is guaranteed, which means that $«_{U6}I$ is an upper bound for the transformed CALE solution matrix due to Lemma 2.1, i.e.

$$\tilde{P} \le «_{U6}I \Leftrightarrow P \le P_{U6} = «_{U6}(S^{*}S)^{-1} = «_{U6}(UE^{2}U^{*})^{-1} = «_{U6}UE^{-2}U^{*}$$

This proves the matrix bound in (2.86). The upper bounds for the maximal eigenvalue and the trace (2.87) and (2.88), respectively, follow easily and this completes the proof of the Lemma.

Remark 2.13 The bounds (2.86)-(2.88) are computable whenever *A* is a stable matrix. As far as we know, these bounds are the first upper estimates for the CALE solution which do not presuppose some additional restrictions imposed on the coefficient matrix *A*. Although the matrix bound (2.86) may be a Hermitian matrix, the rest of the bounds are real. Note, also, that the matrix bound satisfies the inequality $A^{T}P_{U6} + P_{U6}A + Q \le 0$, which means that it satisfies the suppositions of Theorems 2.13 and 2.16 and in accordance with them the respective upper solution estimates can be additionally improved in sense of tightness.

2.6 FURTHER EXTENSION OF VALIDITY SETS

Having at disposal a Lyapunov matrix expressed entirely in terms of the coefficient matrix A is crucial for the derivation of internal upper solution bounds for the CALE, i.e. bounds which do not demand additional procedure for the computation of *T*, in order to achieve a transformed coefficient matrix TAT^{-1} with negative definite symmetric part. The application of the singular value decomposition approach of *A* led to the extension of the conservative set $\boldsymbol{H}^{-}$ and this fact makes possible to get various upper solution bounds in cases when other similar estimates are not computable. An attempt to extend further this result is made now.

Example 2.2 Consider the following stable matrix:

$$A=\begin{bmatrix}-1 & -1\\ 1 & 0.3\end{bmatrix},\quad A_S=\begin{bmatrix}-1 & 0\\ 0 & 0.3\end{bmatrix}$$

Therefore, $A\notin \boldsymbol{H}^-$ and none of the upper solution bounds based on the condition $A\in \boldsymbol{H}^-$ can't be used in this case. Now, consider the matrix sum:

$$A+A^{-1}=\begin{bmatrix}-1 & -1\\ 1 & 0.3\end{bmatrix}+\begin{bmatrix}\frac{3}{7} & \frac{10}{7}\\ -\frac{10}{7} & -\frac{10}{7}\end{bmatrix}=\begin{bmatrix}-\frac{4}{7} & 0\\ 0 & -\frac{79}{70}\end{bmatrix}$$

i.e. the sum of A with its inverse is a negative definite matrix, or this sum belongs to the set of stable matrices with negative definite symmetric parts $\boldsymbol{H}^-$. It will be shown how this fact can be used to extend the set of stable coefficient matrices for which valid upper bounds for the CALE exist.

Theorem 2.19 For any given square matrix A define the set

$$\boldsymbol{H}_E(T,\sim)\equiv \boldsymbol{H}_E \equiv \{T\in \boldsymbol{R}_n, \mathrm{rank}T=n, \sim\,\geq 0 : \exists T,\sim\ \Rightarrow \tilde{A}+\sim\tilde{A}^{-1}\in \boldsymbol{H}^-,\ \tilde{A}=TAT^{-1}\}$$

The following statements hold:

(i) $(T,\sim)\in \boldsymbol{H}_E$ if and only if there exists some nonsingular matrix T and a nonnegative scalar $\sim$ such that

$$P_E=A^TT^TTA+\sim T^TT$$

is a Lyapunov matrix for A

(ii) If $\boldsymbol{H}_L$ denotes the set of Lyapunov matrices P for A, then $\boldsymbol{H}_L\subseteq \boldsymbol{H}_E$

(iii) $\boldsymbol{H}_E(I,1)\subseteq \tilde{\boldsymbol{H}}\subseteq \boldsymbol{H}_E(R_1^{-1/2},\sim)\equiv \boldsymbol{H}_E(R_2^{1/2},\sim)$, where matrices R_1, R_2 are given in (2.2)

(iv) $\boldsymbol{H}_E(I,\sim)\subseteq \boldsymbol{H}_E(R_1^{-1/2},\sim)\equiv \boldsymbol{H}_E(R_2^{1/2},\sim)$

(v) $\boldsymbol{H}_E(T,\sim)\subseteq \boldsymbol{H}_E(\tilde{R}_1^{-1/2}T,\sim)\equiv \boldsymbol{H}_E(\tilde{R}_2^{1/2}T,\sim)$, where matrices $\tilde{R}_1$, $\tilde{R}_2$ are given by

$$\tilde{R}_1=[T(A+\sim A^{-1})(T^TT)^{-1}(A+\sim A^{-1})^TT^T]^{1/2}=(\bar{A}\,\bar{A}^T)^{-1/2}$$

$$\tilde{R}_2=[T^{-T}(A+\sim A^{-1})^T(T^TT)(A+\sim A^{-1})T^{-1}]^{1/2}=(\bar{A}^T\,\bar{A})^{1/2}$$

where $\bar{A}=\tilde{A}+\sim\tilde{A}^{-1}$.

Proof.

(i) Having in mind the definition of the matrix set $\boldsymbol{H}_E$ one gets:

$$0 > (\tilde{A} + \tilde{A}^{-1})^T + \tilde{A} + \sim\tilde{A}^{-1} \Leftrightarrow 0 > \tilde{A}^{2T}\tilde{A} + \tilde{A}^T\tilde{A}^2 + \sim(\tilde{A}^T + \tilde{A})$$

$$= \tilde{A}^T(\tilde{A}^T\tilde{A} + \sim I) + (\tilde{A}^T\tilde{A} + \sim I)\tilde{A}$$

$$= T^{-T}A^TT^T(T^{-T}A^TT^TTAT^{-1} + \sim I) + (T^{-T}A^TT^TTAT^{-1} + \sim I)TAT^{-1}$$

$$< 0 \Leftrightarrow 0 < A^T(A^TT^TTAT + \sim T^TT) + (A^TT^TTA + \sim T^TT)A$$

$$= A^TP_E + P_EA$$

i.e. P_E is a Lyapunov matrix for *A*, which proves the second statement.

(ii) This is obvious, since $\boldsymbol{H_L} \equiv \boldsymbol{H_E}(P^{1/2}, 0)$.

(iii) Consider the singular value decomposition of matrix *A* in (2.1) and (2.2) and suppose that

$$0 > A^T + A + A^{-T} + A^{-1} = F^TR_1 + R_1F + R_1^{-1}F + F^TR_1^{-1} = F^T(R_1 + R_1^{-1}) + (R_1 + R_1^{-1})F$$

Therefore, F is a stable matrix in accordance with Theorem 1.6 and therefore $\boldsymbol{H_E}(I, 1) \subseteq \tilde{\boldsymbol{H}}$. Since F is normal, it is a stable matrix if and only if its symmetric part is negative definite. The following symmetric part inequalities are valid for arbitrary given $\sim\, \geq 0$:

$$0 > [R_1^{1/2}FR_1^{1/2} + \sim R_1^{-1/2}F^TR_1^{-1/2}]_S = [R_1^{-1/2}(R_1F)R_1^{1/2} + \sim R_1^{-1/2}(F^TR_1^{-1})R_1^{1/2}]_S$$

$$= [R_1^{-1/2}AR_1^{1/2} + \sim R_1^{-1/2}A^{-1}R_1^{1/2}]_S \Rightarrow \tilde{\boldsymbol{H}} \subseteq \boldsymbol{H_E}(R_1^{-1/2}, \sim)$$

$$0 > [R_2^{1/2}FR_2^{1/2} + \sim R_2^{-1/2}F^TR_2^{-1/2}]_S = [R_2^{1/2}(FR_2)R_2^{-1/2} + \sim R_2^{1/2}(R_2^{-1}F^T)R_2^{-1/2}]_S$$

$$= [R_2^{1/2}(A)R_2^{-1/2} + \sim R_2^{1/2}(A)R_2^{-1/2}]_S \Rightarrow \tilde{\boldsymbol{H}} \subseteq \boldsymbol{H_E}(R_2^{1/2}, \sim)$$

The statement $\boldsymbol{H_E}(R_1^{-1/2}, \sim) \equiv \boldsymbol{H_E}(R_2^{1/2}, \sim)$ is obvious.

(iv) Suppose that there exists some nonnegative scalar $\sim$, such that

$$0 > (A + A^{-1})^T + \sim(A + \sim A^{-1}) = F^TR_1 + R_1^{-1}F + \sim(R_1F + F^TR_1^{-1}$$

$$= F^T(R_1 + \sim R_1^{-1}) + (R_1 + \sim R_1^{-1})F$$

Then, *F* must be a stable matrix, due to Theorem 1.6, i.e. $A \in \tilde{\boldsymbol{H}}$. According to (iii) $\tilde{\boldsymbol{H}} \subseteq \boldsymbol{H_E}(R_1^{-1/2}, \sim) \equiv \boldsymbol{H_E}(R_2^{1/2}, \sim)$, which proves the statement.

(v) Let there exists some appropriate couple $(T, \sim) \in \boldsymbol{H_E}(T, \sim)$. From the definition of the set in (2.87) this is equivalent to the inequality $[(\tilde{A} + \sim\tilde{A}^{-1})]_S < 0$. Application of Theorem

2.1 for matrix $\bar{A} = \tilde{A} + \mu \tilde{A}^{-1}$ leads to the conclusion that $\bar{A} \in \tilde{\boldsymbol{H}}$ and therefore, $\tilde{R}_1 = (\bar{A}\,\bar{A}^T)^{-1}, \tilde{R}_2 = (\bar{A}^T\,\bar{A})^{1/2}$ are Lyapunov matrices for $\bar{A}$, i.e.

$$0 > [\tilde{R}_1^{-1}(\tilde{A} + \mu \tilde{A}^{-1})]_S \Leftrightarrow 0 > [\tilde{R}_1^{-1/2} T (A + \mu A^{-1}) T^{-1} \tilde{R}_1^{1/2}]_S \Rightarrow (\tilde{R}_1^{-1/2} T, \mu) \in \boldsymbol{H}_E (\tilde{R}_1^{-1/2} T, \mu)$$

Using similar arguments the fact $\boldsymbol{H}_E(T, \mu) \subseteq \boldsymbol{H}_E(\tilde{R}_2^{1/2} T, \mu)$ is proved in a similar way.

Corollary 2.4 The equivalent sets $\boldsymbol{H}_E(R_1^{-1/2}, \mu) \equiv \boldsymbol{H}_E(R_2^{1/2}, \mu)$ are not empty for a given matrix A if and only if some nonnegative scalar μ exists such that

$$P_{E1} = A^T R_1^{-1} A + \mu R_1^{-1}, P_{E2} = A^T R_2 A + \mu R_2 \tag{2.89}$$

are Lyapunov matrices for A.

Proof. It follows from Theorem 2.18, statement (i), for $T = R_1^{-1/2}, T = R_2^{1/2}$, respectively.

Example 2.3 Consider Example 2.1 illustrating the conservativeness of the set $\boldsymbol{H}^-$ in upper bounds derivation. The inverse of matrix A is computed as:

$$A^{-1} = \begin{bmatrix} \alpha\gamma & 0 & -\gamma \\ \gamma & -1 & -\gamma \\ \gamma & 0 & -\gamma \end{bmatrix}; \quad \gamma = \frac{1}{1-\alpha}$$

Matrix A, and consequently, its inverse, are stable for all $\alpha < 1$. It was verified that the symmetric part of A is negative definite for all $\alpha < -0.25$. It is interesting to see whether there exists some $\mu > 0$, such that $(I, \mu) \in \boldsymbol{H}_E$ for $\alpha \geq -0.25$. If this is so, the set of parameter matrices $A(\alpha)$ for which upper solution bounds for the CALE is obviously extended in this case. For $\mu = 0.25$ and $\alpha = 0.7$ the maximal eigenvalue of the symmetric part of $\tilde{A}_S$ was computed as $\lambda_1(\tilde{A}_S) = -0.1206$. According to Theorem 2.18, statement (i), $P_\mu = A^T A + 0.25 I$ is a Lyapunov function for A in this case. In other words, while the bounds based on the restriction $A \in \boldsymbol{H}^-$ are valid only for $\alpha < -0.25$, various computable upper solution bounds can be obtained for all $\alpha < 0.7$.

The result obtained in Theorem 2.19 can be further extended to get even less restrictive conditions.

Theorem 2.20 For any given square matrix A define the set

$$\tilde{\boldsymbol{H}}_E(T, \mu) \equiv \tilde{\boldsymbol{H}}_E \equiv \{T \in \boldsymbol{R}_n, \text{rank} T = n, \mu \geq 0 : \exists T, \mu \Rightarrow \tilde{A} + \mu \tilde{A}^{-1} \in \tilde{\boldsymbol{H}}, \tilde{A} = TAT^{-1}\} \tag{2.90}$$

The following statements hold:

(i) $\boldsymbol{H}_{\boldsymbol{E}} \in \tilde{\boldsymbol{H}}_{\boldsymbol{E}}$

(ii) Denote $\overline{A} = \tilde{A} + \sim \tilde{A}^{-1}$. Then, $(T, \sim) \in \tilde{\boldsymbol{H}}_{\boldsymbol{E}}$ if and only if some nonsingular matrix T and a nonnegative scalar $\sim$ exist such that

$$\tilde{P}_{E1} = A^T T^T \tilde{R}_1^{-1} TA + \sim T^T \tilde{R}_1^{-1} T, \quad \tilde{R}_1 = (\overline{A}\,\overline{A}^T)^{1/2} \tag{2.91}$$

$$\tilde{P}_{E2} = A^T T^T \tilde{R}_2 TA + \sim T^T \tilde{R}_2 T, \quad \tilde{R}_2 = (\overline{A}^T\,\overline{A})^{1/2} \tag{2.92}$$

are Lyapunov matrices for A.

Proof.

(i) Due to Theorem 2.1 we know that $\boldsymbol{H}^- \subseteq \tilde{\boldsymbol{H}}$, i.e., if $(T, \sim) \in \boldsymbol{H}_{\boldsymbol{E}}$, or equivalently, $\overline{A} = \tilde{A} + \sim \tilde{A}^{-1} \in \boldsymbol{H}^-$, then $\overline{A} = \tilde{A} + \sim \tilde{A}^{-1} \in \tilde{\boldsymbol{H}}$, and $(T, \sim) \in \tilde{\boldsymbol{H}}_{\boldsymbol{E}}$, by necessity.

(ii) By the definition of the set in (2.90) and Theorem 2.1 the condition $(T, \sim) \in \tilde{\boldsymbol{H}}_{\boldsymbol{E}}$ is equivalent to the existence of some appropriate T and $\sim$, such that $\tilde{R}_1^{-1}$ and $\tilde{R}_2$ in (2.91) and (2.92), respectively, are Lyapunov matrices for $\overline{A}$, i.e.

$$\begin{aligned}
0 &> \overline{A}^T \tilde{R}_1^{-1} + \tilde{R}_1^{-1} A = (\tilde{A} + \sim \tilde{A}^{-1})^T \tilde{R}_1^{-1} + \tilde{R}_1^{-1} (\tilde{A} + \tilde{A}^{-1}) \\
&= T^{-T} (A + \sim A^{-1})^T T^T \tilde{R}_1^{-1} + \tilde{R}_1^{-1} T (A + \sim A^{-1}) T^{-1} \\
&\Leftrightarrow 0 > (A + \sim A^{-1})^T T^T \tilde{R}_1^{-1} T + T^T \tilde{R}_1^{-1} T (A + \sim A^{-1}) \\
&\Leftrightarrow 0 > (A^2 + \sim I)^T T^T \tilde{R}_1^{-1} TA + A^T T^T \tilde{R}_1^{-1} T (A^2 + \sim I) \\
&= A^T (A^T T^T \tilde{R}_1^{-1} TA + \sim T^T \tilde{R}_1^{-1} T) + (A^T T^T \tilde{R}_1^{-1} TA + \sim T^T \tilde{R}_1^{-1} T) A \\
&= A^T \tilde{P}_{E1} + \tilde{P}_{E1} A
\end{aligned}$$

The proof for the Lyapunov matrix in (2.92) is done in a similar way.

If $T = I$, and $\sim = 0$ then $\overline{A} \equiv A$, $\tilde{R}_1 \equiv R_1$ and $\tilde{R}_2 \equiv R_2$ in (2.2) in this special case.

The set of stable matrices A for which upper CALE solution bounds are computable can be further extended by considering the matrix sum

$$\hat{A} = \overline{A} + \quad \overline{A}^{-1} = (\tilde{A} + \sim \tilde{A}^{-1}) + \quad (\tilde{A} + \sim \tilde{A}^{-1})^{-1}, \quad \tilde{A} = TAT^{-1}, \ \sim, \quad \geq 0$$

2.7 BOUNDS FOR THE CARE SOLUTION

The presented in Chapter I bounds for the solution of the CARE (1.6) admit to characterize briefly the estimation problem as being crucially dependent on the conditions $BB^T > 0$ for the upper and Q > 0 for the lower estimates. Also, the derived bounds include many scalar and matrix parameters involved in various inequalities, which makes them rather complex and not easily computable. Motivated by these awkward facts our main purpose is to get simpler respective bounds which hold under more realistic, i.e. less restrictive conditions.

The main difficulty in deriving bounds for the CARE solution matrix consists in the conservative assumption that either matrix BB^T and/or the state weighting matrix Q are assumed to be positive definite (see e.g. (1.27)-(1.33), (1.36), (1.37), (1.82), (1.84), (1.100), (1.102), (1.103), etc.). Such restrictive assumptions can be compared only to the requirement for negative definiteness of the symmetric part of the coefficient matrix A and positive definiteness of Q for the CALE solution estimation problem.

Having in mind Theorem 1.3, it assumed that the triple (C, A, B) is regular, which guarantees that the solution P of the CARE (1.6) is a positive definite matrix. Also, it is assumed that the control weighting matrix $R = I$ (Remark 1.1).

For any given m x n matrix K consider the CARE (1.6) rewritten as:

$$\begin{aligned}(A-BK)^T P + P(A-BK) &= PBB^T P - K^T B^T P - PBK + K^T K - (Q + K^T K)\\ &= (PB - K^T)(B^T P - K) - (Q + K^T K)\\ &= S - (Q + K^T K) \qquad (2.93)\end{aligned}$$

where $S = (PB - K^T)(B^T P - K)$. Multiplication of both sides of (1.6) by P^{-1} results in:

$$\begin{aligned}P^{-1}(-A-GC)^T + P^{-1}(-A-GC) &= P^{-1}QP^{-1} - P^{-1}C^T G^T - GCP^{-1} + GG^T - (BB^T + GG^T)\\ &= (P^{-1}C^T - G)(CP^{-1} - G^T) - (BB^T + GG^T)\\ &= M - (BB^T + GG^T) \qquad (2.94)\end{aligned}$$

for any given n x m matrix G , where $M = (P^{-1}C^T - G)(CP^{-1} - G^T)$. Using the notations

$$A_U = A - BK \qquad (2.95)$$

$$A_L = -A - GC \qquad (2.96)$$

the modified CAREs (2.93) and (2.94) can be written more compactly as the following Lyapunov-like equations:

$$A_U{}^T P + PA_U = S - (Q + K^T K), S \geq 0 \qquad (2.97)$$

$$P^{-1}A_L{}^T + A_L P^{-1} = M - (BB^T + GG^T), M \geq 0 \qquad (2.98)$$

Theorem 2.21 Consider the matrix in (2.95). Suppose that some appropriate matrix K exists, such that $A_U \in \tilde{\boldsymbol{H}}$. The positive definite solution of the CARE (1.6) has the following upper matrix bounds:

$$P \leq P_{U1} = \alpha_{U1} R_{1U}^{-1}, \quad \alpha_{U1} = \frac{1}{2}\lambda_1\{-(Q + K^T K)(R_{1U}^{-1}A_U)^{-1}]_S\}, \quad R_{1U} = (A_U A_U^T)^{1/2} \qquad (2.99)$$

$$P \leq P_{U2} = \alpha_{U2} R_{2U}, \quad \alpha_{U2} = \frac{1}{2}\lambda_1\{-(Q + K^T K)[(R_{2U}A_U)^{-1}]_S\}, \quad R_{2U} = (A_U^T A_U)^{1/2} \qquad (2.100)$$

Proof. Under the supposition for regularity of the triple (A, B, C) it follows that (A, B) is a stabilizable pair, i.e. some matrix G always exists such that the close loop matrix in (2.95) is a stable one. Also, according to the supposition A_U belongs to the set of matrices with stable orthogonal parts $\tilde{\boldsymbol{H}}$, which is a subset of the set of stable matrices $\boldsymbol{H}$. According to Theorem 2.1, this is possible if and only if R_{1U}^{-1}, R_{2U} , defined in (2.99) and (2.100), respectively, are Lyapunov matrices for the closed loop stable matrix in (2.95), i.e.

$$0 > A_U^T R_{1U}^{-1} + R_{1U}^{-1}A_U = 2(R_{1U}^{-1}A_U)_S \Leftrightarrow 0 > A_U^T R_{2U} + R_{2U}A_U = 2(R_{2U}A_U)_S$$

For the scalars α_{U1}, α_{U2} chosen as above, one gets:

$$\begin{aligned} -(Q + K^T K) &\geq 2\alpha_{U1}(R_{1U}^{-1}A_U)_S \\ &= \alpha_{U1}(A_U^T R_{1U}^{-1} + R_{1U}^{-1}A_U) \\ &= A_U^T P_{U1} + P_{U1}A_U \\ -(Q + K^T K) &\geq 2\alpha_{U2}(R_{2U}A_U)_S \\ &= \alpha_{U2}(A_U^T R_{2U} + R_{2U}A_U) \\ &= A_U^T P_{U2} + P_{U2}A_U \end{aligned}$$

Matrix S (2.93) is positive (semi)-definite and therefore

$$S - (Q + K^T K) \geq -(Q + K^T K) \geq A_U^T P_{U1} + P_{U1}A_U$$

$$S-(Q+K^TK)\geq -(Q+K^TK)\geq A_U^TP_{U2}+P_{U2}A_U$$

Application of Lemma 2.1 with respect to (2.97) results in the upper matrix estimates (2.99) and (2.100), which proves the Theorem.

Theorem 2.22 Consider the matrix in (2.96). Suppose that some appropriate matrix G exists, such that $A_L\in\tilde{\boldsymbol{H}}$. The positive definite solution of the CARE (1.6) has the following lower matrix bounds:

$$P\geq P_{L1}=\frac{1}{\alpha_{L1}}R_{1L}^{-1},\quad \alpha_{L1}=\frac{1}{2}\lambda_1\{-(BB^T+GG^T)(A_LR_{1L})^{-1}]_S\},\quad R_{1L}=(A_LA_L^T)^{1/2}\qquad(2.101)$$

$$P\geq P_{L2}=\frac{1}{\alpha_{L2}}R_{2L},\quad \alpha_{L6}=\frac{1}{2}\lambda_1[-(BB^T+GG^T)(A_LR_{2L}^{-1})_S{}^{-1}],\quad R_{2L}=(A_L^TA_L^T)^{1/2}\qquad(2.102)$$

Proof. The same arguments used to prove Theorem 2.21 are applied. Under the supposition for regularity of the triple (A, B, C) it follows that (C, A) is a detectable pair, i.e. (A^T,C^T) is a stabilizable pair, and some matrix G always exists such that the close loop matrix in (2.96) is a stable one, in this case. Also, it is assumed that A_L belongs to the set of matrices with stable orthogonal parts $\tilde{\boldsymbol{H}}$. According to Theorem 2.1, this is equivalent to

$$0>A_L^TR_{1L}^{-1}+R_{1L}^{-1}A_U\Leftrightarrow 0>A_L^TR_{2L}+R_{2L}A_L$$

where R_{1L}^{-1},R_{2L}, defined in (2.101) and (2.102), obviously are Lyapunov matrices for the closed loop stable matrix in (2.96).This also means that R_{1L},R_{2L}^{-1} are Lyapunov matrices for A_L^T, or

$$0>R_{1L}A_L^T+A_LR_{1L}=2(A_LR_{1L})_S\Leftrightarrow 0>R_{2L}^{-1}A_L^T+A_LR_{2L}^{-1}=2(A_LR_{2L}^{-1})_S$$

If the scalars α_{L1},α_{L2} are chosen as above, and having in mind the notations in (2.101) and (2.102), one gets:

$$\begin{aligned}-(BB^T+GG^T)&\geq 2\alpha_{L1}(A_LR_{1L})_S\\&=\alpha_{L1}(R_{1L}A_L^T+A_LR_{1L})\\&=P_{L1}^{-1}A_L^T+A_LP_{L1}^{-1}\\-(BB^T+GG^T)&\geq 2\alpha_{L2}(A_LR_{2L}^{-1})_S\\&=\alpha_{L2}(R_{2L}^{-1}A_L^T+A_LR_{2L}^{-1})\end{aligned}$$

$$= P_{L2}^{-1} A_L^T + A_L P_{L2}^{-1}$$

Matrix M in (2.94) is positive (semi)-definite and therefore

$$M - (BB^T + GG^T) \geq -(BB^T + GG^T) \geq P_{L1}^{-1} A_L^T + A_L P_{L1}^{-1}$$

$$M - (BB^T + GG^T) \geq -(BB^T + GG^T) \geq P_{L2}^{-1} A_L^T + A_L P_{L2}^{-1}$$

Application of Lemma 2.1 with respect to (2.98) results in the upper matrix estimates for the inverse of the solution matrix:

$$P^{-1} \leq P_{L1}^{-1} = \text{«}_{L1} R_{1L}, \quad P^{-1} \leq P_{L2}^{-1} = \text{«}_{L2} R_{2L}^{-1}$$

These inequalities imply respective lower estimates for the CARE solution P

$$P \geq P_{L1} = (\text{«}_{L1} R_{1L})^{-1}, \quad P \geq P_{L2} = \text{«}_{L2}^{-1} R_{2L}$$

This completes the proof of the Theorem.

Corollary 2.5 The minimal and maximal eigenvalues, the trace of the positive definite solution P of the CARE (1.6) and the performance index J in (1.3) have the following bounds:

$$\lambda_1(P) \leq e_{U1} = \min[\lambda_1(P_{U1}), \lambda_1(P_{U2})], \text{ if } A_U \in \tilde{\boldsymbol{H}} \tag{2.103}$$

$$\lambda_n(P) \geq e_{L1} = \max[\lambda_n(P_{L1}), \lambda_n(P_{L2})], \text{ if } A_L \in \tilde{\boldsymbol{H}} \tag{2.104}$$

$$tr(P) \leq t_{U1} = \min[tr(P_{U1}), tr(P_{U2})], \text{ if } A_U \in \tilde{\boldsymbol{H}} \tag{2.105}$$

$$tr(P) \geq t_{L1} = \max[tr(P_{L1}), tr(P_{L2})], \text{ if } A_L \in \tilde{\boldsymbol{H}} \tag{2.106}$$

Proof. It follows easily having in mind that P_{U1}, P_{U2} are upper matrix bounds for P.

Consider the singular value decomposition of matrix A in (2.1) and (2.2). Then, the singular value decomposition of A_U in (2.95) and A_L in (2.96) is:

$$A_U = R_{1U} \tilde{F} = \tilde{F} R_{2U}, \; R_{1U} = (A_U A_U^T)^{1/2}, \; R_{2U} = (A_U^T A_U)^{1/2} \tag{2.107}$$

$$A_L = R_{1L} \tilde{F} = \tilde{F} R_{2L}, \; R_{1L} = (A_L A_L^T)^{1/2}, \; R_{2L} = (A_L^T A_L)^{1/2} \tag{2.108}$$

Theorem 2.23 Suppose that $A_U \in \tilde{\boldsymbol{H}}$. Then, the maximal eigenvalue and the trace of the CARE solution has the following upper bounds:

$$\lambda_1(P) \leq e_{U2} = \min\{\frac{\lambda_1[(Q + K^T K)(-\tilde{F}_S^{-1})]}{2\sigma_n(A_U)}, \; \frac{\sigma_1(A_U)\lambda_1[(Q + K^T K)(-R_{2U}\tilde{F}_S R_{2U})^{-1}]}{2}\} \tag{2.109}$$

$$tr(P) \le t_{U2} = \min\{\frac{tr[\tilde{R}_{1U}(Q+K^T K)}{-2\lambda_1(R_{1U}\tilde{F}_S R_{1U})}, \frac{tr[R_{2U}^{-1}(Q+K^T K)}{-2\lambda_1(\tilde{F}_S)}\} \qquad (2.110)$$

$$tr(P) \le t_{U3} = \min\{\frac{1}{\sigma_n(A_U)}\sum_{i=1}^{n}\frac{\lambda_i[R_{1U}(Q+K^T K)]}{-2\lambda_1(R_{1U}\tilde{F}_S)}, \sigma_1(A_U)\sum_{i=1}^{n}\frac{\lambda_i[R_{2U}^{-1}(Q+K^T K)}{-2\lambda_1(R_{2U}\tilde{F}_S)}\} \qquad (2.111)$$

If $A_L \in \tilde{\boldsymbol{H}}$, then the minimal eigenvalue of the CARE solution has the lower bound:

$$\lambda_n(P) \ge e_{L2} = \max\{\frac{2}{-\sigma_1(A_L)\lambda_n[(BB^T+GG^T)(R_{1L}\tilde{F}_S R_{1L})^{-1}]}, \frac{2\sigma_n(A_L)}{-\lambda_n[(BB^T+GG^T)(\tilde{F}_S)^{-1}]}\} \qquad (2.112)$$

Proof. Consider the Lyapunov-type equation (2.97) and the singular value decomposition in (2.107). If $A_U \in \tilde{\boldsymbol{H}}$, then application of the CALE eigenvalue bounds (2.17) for (2.97) results in the following upper eigenvalue bounds for the CARE solution:

$$\lambda_1(P) \le \min(\frac{\lambda_1[(-S+Q+K^T K)(-\tilde{F}_S^{-1})]}{2\sigma_n(A_U)}, \frac{\sigma_1(A_U)\lambda_1[(-S+Q+K^T K)(-R_{2U}\tilde{F}_S R_{2U})^{-1}]}{2})$$

It is well known that

$$\lambda_1(XY) \le \lambda_1(ZY), \quad tr(XY) \le tr(ZY)$$

for arbitrary symmetric matrices *X*, *Y* and *Z*, such that $0 \le X \le Z$ and $Y \ge 0$, and the bound in (2.109) follows since $S \ge 0$ and therefore, $-S+(Q+K^T K) \le Q+K^T K$.

In a similar way, the trace bounds (2.110) and (2.111) for the CARE are easily obtained when the bounds (2.4), (2.6) and (2.8) for the CALE (1.2) are applied for (2.97).

By making use of (2.108), the CARE (2.98) can be rewritten and estimated as

$$P^{-1}\tilde{F}^T R_{1L} + R_{1L}\tilde{F}P^{-1} \ge -(BB^T+GG^T)$$

$$P^{-1}R_{2L}\tilde{F}^T + \tilde{F}R_{2L}P^{-1} \ge -(BB^T+GG^T)$$

Pre- and post-multiplication by $R_{1L}{}^{-1/2}$ and $R_{2L}{}^{1/2}$ of both inequalities, respectively, results in:

$$(R_{1L}^{-1/2}P^{-1}R_{1L}^{-1/2})(R_{1L}^{1/2}\tilde{F}^T R_{1L}^{1/2}) + (R_{1L}^{1/2}\tilde{F}R_{1L}^{1/2})(R_{1L}^{-1/2}P^{-1}R_{1L}^{-1/2}) \ge -R_{1L}^{-1/2}(BB^T+GG^T)R_{1L}^{-1/2}$$

$$(R_{2L}^{1/2}P^{-1}R_{2L}^{1/2})(R_{2L}^{1/2}\tilde{F}^T R_{2L}^{1/2}) + (R_{2L}^{1/2}\tilde{F}R_{2L}^{1/2})(R_{2L}^{1/2}P^{-1}R_{2L}^{1/2}) \ge -R_{2L}^{1/2}(BB^T+GG^T)R_{2L}^{1/2}$$

Using the notations

$$R_{1L}^{1/2}\tilde{F}R_{1L}^{1/2} = F_1,\ R_{1L}^{-1/2}P^{-1}R_{1L}^{-1/2} = P_1,\ R_{1L}^{-1/2}(BB^T+GG^T)R_{1L}^{-1/2} = Q_1$$

$$R_{2L}^{1/2}\tilde{F}R_{2L}^{1/2} = F_2,\ R_{2L}^{1/2}P^{-1}R_{2L}^{1/2} = P_2,\ R_{2L}^{1/2}(BB^T+GG^T)R_{2L}^{1/2} = Q_2$$

the above inequalities can be rewritten in a compact form as:

$$P_1F_1^T + F_1P_1 \geq -Q_1$$

$$P_2F_2^T + F_2P_2 \geq -Q_2$$

respectively. If the supposition $A_L \in \tilde{\boldsymbol{H}}$ holds, than $\tilde{F}_S < 0$ and therefore, the symmetric parts F_{1S}, F_{2S} of matrices F_1, F_2 are negative definite, as well. Let x be an eigenvector corresponding to the maximal eigenvalue of P_1, i.e. $P_1x = \lambda_1(P_1)x$ and consider (2.98). It follows that

$$x^TQ_1x \geq 2x^T[(-F_{1S})]x\lambda_1(P_1) \Rightarrow \lambda_1(P_1) \leq \frac{y^T(-F_{1S})^{-1/2}Q_1(-F_{1S})^{-1/2}y}{2y^Ty} \leq \frac{\lambda_1[Q_1(-F_{1S})^{-1}]}{2}$$

where $y = (-F_{1S})^{1/2}x$. The maximal eigenvalue of matrix P_1 can be estimated from below as:

$$\lambda_1(P_1) = \lambda_1(R_{1L}^{-1}P^{-1}) \geq \lambda_n(R_{1L}^{-1})\lambda_1(P^{-1}) = \frac{1}{\lambda_1(R_{1L})\lambda_n(P)} = \frac{1}{\pm_1(A_L)\lambda_n(P)}$$

i.e.

$$\frac{1}{\pm_1(A_L)\lambda_n(P)} \leq \frac{\lambda_1[Q_1(-F_{1S})^{-1}]}{2} = \frac{-\lambda_n(Q_1F_{1S}^{-1})}{2}$$

$$= \frac{-\lambda_n[R_{1L}^{-1/2}(BB^T + GG^T)R_{1L}^{-1/2}(R_{1L}^{1/2}F_{1S}R_{1L}^{1/2})^{-1}]}{2}$$

$$= \frac{-\lambda_n[(BB^T + GG^T)(R_{1L}\tilde{F}_SR_{1L})^{-1}]}{2}$$

which proves the first bound in (2.112). The second one is obtained in a similar way. This completes the proof of the Theorem.

Remark 2.14 If there exists some matrix G such that matrix A_L in (2.96) belongs to the set $\tilde{\boldsymbol{H}}$, then positive definiteness of the lower matrix bounds (2.101), (2.102) is guaranteed. Also, the minimal solution eigenvalue lower bounds (2.104) and (2.112) are not trivial, in this case. The derived upper and lower CARE solution bounds (2.99)-(2.106), (2.109)-(2.112) do not require the usual validity conditions, i.e. $BB^T > 0$ and $Q > 0$. These bounds are based on two assumptions:

(i) the triple (A, B, C) is regular, which guarantees positive definiteness of the solution P

(ii) the close loop matrices in (2.95) and (2.96) belong to the set $\tilde{\boldsymbol{H}}$.

In fact, (i) is not a restriction at all. The assumption (ii) is less conservative than the assumption (1.79) according to which some positive scalar $\sim$ exists, such that $A^T + A - 2\sim BB^T < 0$. Due to Theorem 2.1 this is possible only if $A - \sim BB^T \in \tilde{\boldsymbol{H}}$, which means that $A_U = A - BK \in \tilde{\boldsymbol{H}}$, $K = \sim B^T$, and all upper bounds based on the singular value decomposition approach are valid in this case, as well. This fact can be illustrated by the following simple example.

Example 2.4 Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}, \quad BB^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A^T + A - 2\sim BB^T = \begin{bmatrix} 2 & 1 \\ -1 & 1-2\sim \end{bmatrix}$$

Obviously, the condition $A^T + A - 2\sim BB^T < 0$ can't be satisfied and the upper matrix bounds (1.80), (1.81) are not valid, in this case. Now, consider the matrix

$$A_U = A - \sim BB^T = \begin{bmatrix} 1 & 2 \\ -1 & -\sim \end{bmatrix}$$

which is stable for all $1 < \sim < 2$. Let $\sim = 1.5$. The unitary matrix $\tilde{F}$ in (2.109) is stable, i.e. $A_U \in \tilde{\boldsymbol{H}}$ and all bounds due to Theorems 2.22, 2.23 and Corollary 2.3 are computable.

Remark 2.15 The lower and the upper matrix bounds obtained for the CARE are both based on the fact that if $A_U \in \tilde{\boldsymbol{H}}$ and $A_L \in \tilde{\boldsymbol{H}}$, symmetric positive definite matrices M_U and M_L exist, such that

$$A_U^T M_U + M_U A_U + Q_U \le 0, \ Q_U = Q + K^T K \tag{2.113}$$

$$M_L A_L^T + A_L M_L + Q_L \le 0, \ Q_L = BB^T + GG^T \tag{2.114}$$

where, $M_U = P_{U1}, P_{U2}$ and $M_L = P_{L1}^{-1}, P_{L2}^{-1}$ are defined in (2.99), (2.100) and (2.101), (2.102), respectively. It was proved, that M_U and M_L are upper matrix bounds, for the CARE solution and its inverse, respectively, or $M_L^{-1} = P_{L1}, P_{L2}$ are lower bounds for P in (1.6). The inequalities (2.113) and (2.114) play an essential role in the improvement of both lower and upper CARE matrix bounds.

Theorem 2.24 Suppose that the inequalities (2.113), (2.114) are satisfied. Then, the solution of the CARE can be bounded as follows:

$$P \ge L_i^{-1}, \quad \forall i = 0,1,2,.. \quad L_i = \tilde{A}_L^i M_L (\tilde{A}_L^i)^T + \sum_{j=0}^{i-1} \tilde{A}_L^j \tilde{Q}_L (\tilde{A}_L^j)^T, \forall i = 0,1,2,.. \quad (2.115)$$

and

$$P \le U_i = (\tilde{A}_U^i)^T M_U \tilde{A}_U^i + \sum_{j=0}^{i-1} (\tilde{A}_U^j)^T \tilde{Q}_U \tilde{A}^j, \forall i = 0,1,2,.. \quad (2.116)$$

where

$$\tilde{A}_L = (A_L - I)^{-1}(A_L + I), \tilde{Q}_L = 2(A_L - I)^{-1} Q_L (A_L - I)^{-T}, Q_L = BB^T + GG^T \quad (2.117)$$

$$\tilde{A}_U = (A_U - I)^{-1}(A_U + I), \tilde{Q}_U = 2(A_U - I)^{-T} Q_U (A_U - I)^{-1}, Q_U = Q + K^T K \quad (2.118)$$

$$M_U = P_{U1}, P_{U2}, \quad M_L = P_{L1}, P_{L2} \quad (2.119)$$

Further,

$$L_i^{-1} \ge L_{i-1}^{-1}, \quad \forall i = 1, 2,..., \quad (2.120)$$

$$U_i \le U_{i-1}, \quad \forall i = 1, 2,... \quad (2.121)$$

where P_{U1}, P_{U2} and P_{L1}, P_{L2} are given in (2.99), (2.100) and (2.101), (2.102), respectively.

Proof. Consider the equivalent representation of the CALE (1.2) in (1.93). Let $\sim = 1$. In a similar way, the Lyapunov-like equations (2.97) and (2.98) can be represented in an equivalent DALE-type form as:

$$P^{-1} = \tilde{A}_L P^{-1} \tilde{A}_L^T - \tilde{M} + \tilde{Q}_L \le \tilde{A}_L P^{-1} \tilde{A}_L^T + \tilde{Q}_L$$

$$P = \tilde{A}_U^T P \tilde{A}_U - \tilde{S} + \tilde{Q}_U \le \tilde{A}_U^T P \tilde{A}_U + \tilde{Q}_U$$

where matrices $\tilde{A}_L$, $\tilde{A}_U$, $\tilde{Q}_L$, $\tilde{Q}_U$ are given in (2.117)-(2.119) and $\tilde{M}, \tilde{S}$ are arbitrary positive (semi)-definite matrices.

The proof is based on Theorem 2.12 and the simple fact that if X and Y are some symmetric matrices such that

$$\tilde{A}_L P^{-1} \tilde{A}_L^T + \tilde{Q}_L \le X \ , \quad \tilde{A}_U^T P \tilde{A}_U + \tilde{Q}_U \le Y$$

then

$$P^{-1} \le X \Leftrightarrow P \ge X^{-1}, \quad P \le Y$$

If the matrix inequalities (2.113) and (2.114) are satisfied, then $M_U = P_{U1}, P_{U2}$ and $M_L =$ P_{L1}^{-1}, P_{L2}^{-1} are upper bounds for the CARE solution P and its inverse P^{-1} , respectively. Therefore, the suppositions of Theorem 2.12 are met with respect to (2.97), (2.98). By means of an appropriate replacement of matrices $A, \tilde{A}, \tilde{Q}, P_L, P_U$ in (2.37) by matrices $(A_U, A_L), (\tilde{A}_U, \tilde{A}_L), (\tilde{Q}_U, \tilde{Q}_L), M_L^{-1}, M_U$, the inequalities in (2.115) and (2.116) actually correspond to (2.36). The same refers to (2.121) and (2.39). Since matrices L_i (2.15) are upper bounds for P^{-1} satisfying in accordance with Theorem 2.13 the inequalities $L_i \leq L_{i-1}, \forall i = 0, 1, 2,...,$ the lower matrix bounds (2.114) for P satisfy the inequalities (2.120)., which proves the statements.

Remark 2.16 Theorem 2.12 can be applied to get tighter matrix bounds for the CALE solution. In a similar way, Theorem 2.23 suggests a possibility to improve both lower and upper matrix bounds for the CARE solution. The only restriction in its application concerns the existence of matrices *K* and *G*, such that the matrices in (2.95) (for the upper bound) and (2.96) (for the lower bound) belong to the set $\tilde{\boldsymbol{H}}$. This restriction is less conservative than the usual and rather not realistic assumptions for positive definiteness of matrices BB^T and *Q*. If this is so, there always exist positive scalars and *q*, such that the matrices

$$A_U = A - BK = A - \ BB^T, K = \ B^T$$

$$A_L = -A - GC = -A - qC^T C = -A - qQ, G = qC^T, C^T C = Q$$

belong to the set $\tilde{\boldsymbol{H}}$ and even to the set $\boldsymbol{H}^{-}$.

Corollary 2.6 Suppose that the inequalities (2.113), (2.114) are satisfied. Then, the extremal eigenvalues and the trace of the CARE solution can be bounded as follows:

$$\text{a}_n(P) \geq e_{L2,i} = \text{a}_n(L_i^{-1}), \quad e_{L2,i} \geq e_{L2,i-1}, \quad \forall i = 1, 2... \tag{2.122}$$

$$\text{a}_1(P) \leq e_{U3,i} = \text{a}_1(U_i), \quad e_{U3,i} \leq e_{U3,i-1} \quad \forall i = 1, 2... \tag{2.123}$$

$$tr(L_i^{-1}) = t_{L2,i} \leq tr(P) \leq t_{U4,i} = tr(U_i), \quad t_{L2,i} \geq t_{L2,i-1}, \quad t_{U4,i} \leq t_{U4,i-1} \quad \forall i = 1, 2,... \tag{2.124}$$

where matrices L_i, U_i, i = 1, 2,… are given in (2.115) and (2.16), respectively.

Proof. The bounds (2.122)-(2.124) follow easily from Theorem 2.24.

Theorem 2.25 Let l_i , $i = 2,..,n$, denotes some nonnegative lower bound for the *i*-th eigenvalue of the CARE solution *P* in (1.6). The solution trace of the CARE has the following bounds:

$$\frac{\frac{1}{2}tr(Q_U)+¥_2(A_{US},P)}{-ª_n(A_{US})}=t_{L3}\le tr(P)\le t_{U5}\,\frac{\frac{1}{2}tr(Q_U)-¥_1(A_{US},P)}{-ª_1(A_{US})} \qquad (2.125)$$

$$\frac{\frac{1}{2}tr(R_{1U}Q_U)+¥_2(R_{1U}F_{US}R_{1U},P)}{-ª_n(R_{1U}F_{US}R_{1U})}\le t_{L4}=tr(P)\le t_{U6}=\frac{\frac{1}{2}tr(R_{1U}Q_U)-¥_1(R_{1U}F_{US}R_{1U},P)}{-ª_1(R_{1U}F_{US}R_{1U})} \qquad (2.126)$$

$$\frac{\frac{1}{2}tr(R_{2U}^{-1}Q_U)+¥_2(F_{US},P)}{-ª_n(F_{US})}\le t_{L5}=tr(P)\le t_{U7}=\frac{\frac{1}{2}tr(R_{2U}^{-1}Q_U)-¥_1(F_{US},P)}{-ª_1(F_{US})} \qquad (2.127)$$

where the upper estimates hold if $A_U \in \boldsymbol{H}^-$ (2.125), $A_U \in \tilde{\boldsymbol{H}}$ ((2.126) and (2.127)), and the scalars $¥_1(X,Y)$, $¥_2(X,Y)$ are defined in (2.48).

Proof. Consider the modified CARE (2.97) and the associated with it matrix inequality:

$$A_U{}^T P + PA_U \ge -Q_U \qquad (2.128)$$

Having in mind the singular value decomposition (2.109) of the stable matrix A_U (2.95) the above inequality can be rewritten as:

$$\tilde{F}^T R_{1U} P + PR_{1U}\tilde{F} \ge -Q_U$$

$$R_{2U}\tilde{F}^T P + P\tilde{F}R_{2U} \ge -Q_U$$

Pre-multiplication of the first inequality by R_{1U} and of the second one by R_{2U}^{-1} results in:

$$R_{1U}\tilde{F}^T R_{1U}P + R_{1U}PR_{1U}\tilde{F} \ge -R_{1U}Q_U \qquad (2.129)$$

$$\tilde{F}^T P + R_{2U}^{-1}P\tilde{F}R_{2U} \ge -R_{2U}^{-1}Q_U \qquad (2.130)$$

Application of the trace operator to both sides of (2.125)-(2.127) leads to the following respective scalar inequalities:

$$tr(Q_U) \ge -2tr(A_{US}P) \qquad (2.131)$$

$$tr(R_{1U}Q_U) \ge -2tr(R_{1U}\tilde{F}_S R_{1U}P) \qquad (2.132)$$

$$tr(R_{2U}^{-1}Q_U) \ge -2tr(\tilde{F}P) \qquad (2.133)$$

The upper trace bounds for the CARE solution follow easily when the used to improve trace bounds for the CALE solution Theorem 2.14 is accordingly applied. E.g., the right-hand side of (2.128) can be evaluated as in (2.54) and (2.55), having in mind that here P denotes the CARE solution, and instead of A_S we consider now A_{US}. The same refers to the rest of the trace bounds by replacing $F \to \tilde{F}, R_1 \to R_{1U}, R_2 \to R_{2U}, Q \to Q_U$ and using (2.132) and (2.133).

Theorems 2.23 and 2.24 show how lower and upper bounds for the CARE solution can be used to improve matrix and trace bounds in sense of tightness, respectively. The statements are based on the modified CARE's (2.97) and (2.98). Now, a based on the CARE (1.6) similar approach, is suggested to derive lower bound for the solution trace.

Theorem 2.26 Suppose that nonnegative scalars $l_i \leq ª_i(P), i = 2,..,n$ exist. The trace of the CARE solution has the following lower bound

$$tr(P) \geq \frac{b + \sqrt{b^2 + ac}}{a} \qquad (2.134)$$

where

$$a = ±_1^2(B), b = ª_n(A_S), c = ¥_1(BB^T, P^2) + ¥_2(A_S, P) + 2±_1^2(B)¥_3 + tr(Q) \qquad (2.135)$$

The scalars $¥_1(BB^T, P^2), ¥_2(A_S, P)$ are defined in (2.48) and $¥_3 = \sum_{\substack{i,j=1 \\ i<j}}^{n} l_i l_j$.

Proof. Consider the CARE (1.6) and the associated with it trace equality

$$2tr[(A_S)P] + tr(Q) = tr(BB^T P^2) \qquad \mathbf{(}2.136)$$

Having in mind (2.47) and (2.48) the next trace estimations can be done:

$$\begin{aligned} tr[(A_S)P] &\geq \sum_{i=1}^{n} ª_{n-i+1}(A_S)ª_i(P) \\ &= ª_n(A_S)tr(P) + \sum_{i=2}^{n}[ª_{n-i+1}(A_S) - ª_n(A_S)]ª_i(P) \\ &\geq ª_n(A_S)tr(P) + \sum_{i=2}^{n}[ª_{n-i+1}(A_S) - ª_n(A_S)]l_i \\ &\geq ª_n(A_S)tr(P) + ¥_2(A_S, P) \end{aligned} \qquad (2.137)$$

$$\begin{aligned} tr(BB^T P^2) &\leq \sum_{i=1}^{n} ª_i(BB^T)ª_i(P^2) \\ &= ª_1(BB^T)tr(P^2) + \sum_{i=2}^{n}[ª_i(BB^T) - ª_1(BB^T)]ª_i(P^2) \\ &\leq ª_1(BB^T)tr(P^2) - \sum_{i=2}^{n}[ª_1(BB^T) - ª_i(BB^T)]l_i^2 \end{aligned}$$

$$= ±_1^2(B)tr(P^2) - \sum_{i=2}^{n}[±_1^2(B) - ±_i^2(B)]l_i^2$$

$$= ±_1^2(B)tr(P^2) - ¥_1(BB^T, P^2) \qquad (2.138)$$

Additionally,

$$tr(P^2) = \sum_{i=1}^{n} a_i^2(P) = [\sum_{i=1}^{n} a_i(P)]^2 - 2\sum_{\substack{i,j=1\\ i<j}}^{n} a_i(P)a_j(P)$$

$$= [tr(P)]^2 \; - 2\sum_{\substack{i,j=1\\ i<j}}^{n} a_i(P)a_j(P)$$

$$\leq [tr(P)]^2 \; - 2\sum_{\substack{i,j=1\\ i<j}}^{n} l_i l_j$$

$$= [tr(P)]^2 - 2¥_3 \qquad (2.139)$$

By making use of (2.136)-(1.139) and using the notation $tr(P) = t$, one finally gets the following quadratic in the solution trace inequality

$$at^2 - 2bt - c \geq 0$$

The parameters *a*, *b*, and *c* are given in (2.135). This completes the proof of the bound (2.134).

Remark 2.17 If the eigenvalue bounds $l_i \leq a_i(P), i = 2,..,n$ for the solution *P* are not taken into account the bound (2.134) is equal to the lower trace estimate (1.35). In the opposite case, (2.135) is tighter than (1.35) for all appropriate matrices *A*, *B* and *Q*. This fact illustrates the advantage of using available solution bounds in the estimation process.

Finally, lower and upper bounds for the CARE solution can be applied to estimate the performance index *J* (1.7). For any lower and upper matrix bound one has

$$x_0^T P_L x_0 = J_L \leq J \; \leq J_U = x_0^T P_U x_0 \;\; \forall x_0 \neq 0$$

2.8 THE CASE $TAT^{-1} \in \boldsymbol{H}^-$

Motivated by the fact that upper bounds validity depends crucially on the conservative condition $A \in \boldsymbol{H}^-$, an attempt to overcome this difficult problem in solution estimation was firstly made in 1997. It was suggested in [30] to replace the condition for negative definiteness of the symmetric part of the coefficient matrix A in (1.2), with a less restrictive one. i.e., $TAT^{-1} \in \boldsymbol{H}^{-}$, for some nonsingular matrix T which needs to be computed. The nonsingular transformation preserves the eigenvalues of matrix A, but it changes the eigenvalues of its symmetric part. Finally, due to Theorem 1.6, the desired property for the transformed matrix A can always be achieved. As a result, the upper trace and maximal eigenvalue solution bounds (1.53)-(1.55) were obtained.

Our purpose is to show that if such an approach is applied, respective based on the singular value decomposition approach computable upper bounds for the CALE can always be obtained, as well.

Suppose that $TAT^{-1} = \tilde{A} \in \boldsymbol{H}^{-}$, for some matrix T and consider the modified CALE (1.52). All upper scalar and matrix upper bounds for the CALE solution hold, if $A \in \tilde{\boldsymbol{H}}$. In the opposite case, i.e. $A \notin \tilde{\boldsymbol{H}}$, one has $\tilde{A} \in \tilde{\boldsymbol{H}}$, in accordance with Theorem 2.1. Having in mind (2.1) and (2.2) the transformed matrix can be represented as:

$$\tilde{A} = \tilde{U}\tilde{\Sigma}\tilde{V}^T = \tilde{R}_1\tilde{F} = \tilde{F}\tilde{R}_2, \ \tilde{R}_1 = (\tilde{A}\tilde{A}^T)^{1/2}, \quad \tilde{R}_2 = (\tilde{A}^T\tilde{A})^{1/2}$$

Then, the CALE (1.52) is rewritten as:

$$\tilde{F}^T\tilde{R}_1\tilde{P} + \tilde{P}\tilde{R}_1\tilde{F} = -\tilde{Q}$$

or,

$$\tilde{R}_2\tilde{F}^T\tilde{P} + \tilde{P}\tilde{F}\tilde{R}_2 = -\tilde{Q}$$

where the transformed solution $\tilde{P}$ and right-hand side matrix $\tilde{Q}$ are given in (1.52). Various based on the singular value decomposition approach lower and upper, scalar and matrix bounds can be derived for the solution of the transformed CALEs and using some suitable estimation techniques they can be used to obtain respective bounds for the CALE solution (1.2).

As it was already said, such a transformation based estimation approach may result in serious problems concerning the computational complexity. This is the main reason why, our main purpose is to extend the bounds validity via the derivation of internal bounds (Definition 1.4).

CHAPTER THREE

BOUNDS FOR THE DISCRETE-TIME EQUATIONS

3.1 THE SINGULAR VALUE DECOMPOSITION APPROACH

As far as upper bounds are concerned, the main difficulty in the estimation problem for the DALE solution consists in the rather conservative assumption that the spectral norm of the coefficient matrix in (1.9) is less than one. It may be compared to the assumption that the symmetric part of the stable in continuous-time sense matrix A is negative definite. An approach to overcome to a certain extent this shortcoming for the CALE bounds was suggested in the previous chapter. The extension of the set of stable in discrete-time sense matrices for which upper bounds for the DALE solution are computable, will be investigated now.

Consider the singular value decomposition of an arbitrary n x n matrix A in (2.1) and (2.2). If A is stable in the continuous-time sense, then positive definiteness of matrices $R_1 = (AA^T)^{1/2}, R_2 = (A^T A)^{1/2}$ is guaranteed. Since a stable in the discrete-time sense may be singular, then R_1, R_2 are positive (semi)-definite matrices, in the general case. Having in mind this and (2.1) the singular value decomposition of the coefficient matrix in the DALE (1.9) is

$$A = U\Sigma V^T, \quad UU^T = VV^T = I \tag{3.1}$$

$$\Sigma = \begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \tag{3.2}$$

where $r = rank(A) \le n$ and the diagonal matrix Ξ_r contains the r positive singular values of A. Consider the accordingly partitioned orthogonal matrix

$$F = V^T U = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad F \in \boldsymbol{R}_r, F_{12} \in \boldsymbol{R}_{r,n-r}, F_{21} \in \boldsymbol{R}_{n-r,r}, F_{22} \in \boldsymbol{R}_{n-r} \qquad (3.3)$$

Define the matrix sets:

$$\boldsymbol{S} \equiv \{A,\ A \in \boldsymbol{R}_n : \det(A - \lambda I) = 0 \Rightarrow |\lambda| < 1\}$$

$$\boldsymbol{S}^1 \equiv \{A,\ A \in \boldsymbol{R}_n : \sigma_1(A) < 1\}$$

$$\tilde{\boldsymbol{S}} \equiv \{A,\ A \in \boldsymbol{R}_n : \sigma_1(\Xi_r^{1/2} F_{11} \Xi_r^{1/2}) < 1\}$$

where the r x r matrices Ξ_r and F_{11} are given in (3.2) and (3.3), respectively. $\boldsymbol{S}$ is the set of stable matrices, $\boldsymbol{S}^1$ is the set of matrices with spectral norms less than one and $\tilde{\boldsymbol{S}}$ is a new matrix set, which shall be extensively used to get less restrictive validity conditions for the upper solution bounds.

Define the matrices

$$\Phi_1 = U \tilde{\Sigma}_1 U^T, \quad \tilde{\Sigma}_1 = \begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & \gamma_1^2 I_{n-r} \end{bmatrix}, \quad \gamma_1 > 0 \qquad (3.4)$$

$$\Phi_2 = V \tilde{\Sigma}_2 V^T, \quad \tilde{\Sigma}_2 = \begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & \gamma_2^2 I_{n-r} \end{bmatrix}, \quad \gamma_2 > 0, \qquad (3.5)$$

Theorem 3.1 The following statements hold:

(i) $\boldsymbol{S}^1 \subseteq \tilde{\boldsymbol{S}}$

(ii) $A \in \tilde{\boldsymbol{S}}$, if and only if some positive scalars γ_1, γ_2 exist, such that Φ_1^{-1}, Φ_2 are Lyapunov matrices for A.

Proof Suppose, that $A \in \boldsymbol{S}^1$, or equivalently, $\sigma_1(A) = \sigma_1(\Xi_r) < 1$. Since F is an orthogonal matrix, its spectral norm is equal to one, which means that $\sigma_1(F_{11}) \leq 1$, by necessity. The spectral norm of matrix $\Xi_r^{1/2} F_{11} \Xi_r^{1/2}$ can be evaluated as follows:

$$\sigma_1(\Xi_r^{1/2} F_{11} \Xi_r^{1/2}) \leq \sigma_1(\Xi_r)\sigma_1(F_{11}) \leq \sigma_1(\Xi_r) < 1$$

Therefore, $A \in \boldsymbol{S}^1$ only if $A \in \tilde{\boldsymbol{S}}$, which proves the first statement.

Suppose that some positive scalar γ_1 exist, such that (3.4) is a Lyapunov matrix for A, i.e

$$0 > A^T\Phi_1^{-1}A - \Phi_1^{-1} \Leftrightarrow I > \Phi_1^{1/2}A^T\Phi_1^{-1}A\Phi_1^{1/2} \Leftrightarrow 1 > \pm_1(\tilde{A}_1),\ \tilde{A}_1 = \Phi_1^{-1/2}A\Phi_1^{1/2}$$

Taking into account (3.2)-(3.4) one gets:

$$\tilde{A}_1 = U\tilde{\Sigma}_1^{-1/2}U^TU\Sigma V^TU\tilde{\Sigma}_1^{1/2}U^T = U\tilde{\Sigma}_1^{-1/2}\Sigma F\tilde{\Sigma}_1^{1/2}U^T$$

$$\Rightarrow 1 > \pm_1(\tilde{A}_1) = \pm_1(\tilde{\Sigma}_1^{-1/2}\Sigma F\tilde{\Sigma}_1^{1/2})$$

$$= \pm_1(\begin{bmatrix} \Xi_r^{-1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1^{-1}I_{n-r} \end{bmatrix}\begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} F \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1 I_{n-r} \end{bmatrix})$$

$$= \pm_1(\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} F \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1 I_{n-r} \end{bmatrix})$$

$$= \pm_1(\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1 I_{n-r} \end{bmatrix})$$

$$= \pm_1(\begin{bmatrix} \Xi_r^{1/2}F_{11}\Xi_r^{1/2} & \circ_1\Xi_r^{1/2}F_{12} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}) = \pm_1(\Gamma_1)$$

Since $\pm_1(\Gamma_1) < 1 \Leftrightarrow \text{ª}_1(\Gamma_1^T\Gamma_1) < 1$, one finally has

$$1 > \text{ª}_1(\Gamma_1\Gamma_1^T) = \text{ª}_1(\begin{bmatrix} \Xi_r^{1/2}F_{11}\Xi_r^{1/2} & \circ_1\Xi_r^{1/2}F_{12} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}\begin{bmatrix} \Xi_r^{1/2}F_{11}^T\Xi_r^{1/2} & 0_{r,n-r} \\ \circ_1 F_{12}^T\Xi_r^{1/2} & 0_{n-r} \end{bmatrix})$$

$$= \text{ª}_1(\begin{bmatrix} \Xi_r^{1/2}F_{11}\Xi_r F_{11}^T\Sigma_r^{1/2} + \circ_1^2\Xi_r^{1/2}F_{12}F_{12}^T\Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}) \qquad (3.6)$$

$$\Rightarrow \text{ª}_1(\Xi_r^{1/2}F_{11}\Xi_r F_{11}^T\Xi_r^{1/2}) = \pm_1^2(\Xi_r^{1/2}F_{11}\Xi_r^{1/2}) < 1$$

$$\Leftrightarrow \pm_1(\Xi_r^{1/2}F_{11}\Xi_r^{1/2}) < 1$$

which proves the necessity part of statement (ii) for Φ_1^{-1}.

Now, consider the case in which Φ_2 (3.5) is Lyapunov matrix for *A*, i.e

$$0 > A^T\Phi_2 A - \Phi_2 \Leftrightarrow I > \Phi_2^{-1/2}A^T\Phi_2 A\Phi_2^{-1/2} \Leftrightarrow 1 > \pm_1(\tilde{A}_2),\ \tilde{A}_2 = \Phi_2^{1/2}A\Phi_2^{-1/2}$$

Having in mind (3.2), (3.3) and (3.5) one gets:

$$\tilde{A}_2 = V\tilde{\Sigma}_2^{1/2}V^TU\Sigma V^TV\tilde{\Sigma}_2^{-1/2}V^T = V\tilde{\Sigma}_2^{1/2}F\Sigma\tilde{\Sigma}_2^{-1/2}V^T$$

$$\Rightarrow 1 > \pm_1(\tilde{A}_2) = \pm_1(\tilde{\Sigma}_2^{1/2}F\Sigma\tilde{\Sigma}_2^{-1/2})$$

$$= \pm_1 (\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_2 I_{n-r} \end{bmatrix} F \begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \begin{bmatrix} \Xi_r^{-1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_2^{-1} I_{n-r} \end{bmatrix})$$

$$= \pm_1 (\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_2 I_{n-r} \end{bmatrix} F \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix})$$

$$= \pm_1 (\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_2 I_{n-r} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix})$$

$$= \pm_1 (\begin{bmatrix} \Xi_r^{1/2} F_{11} \Xi_r^{1/2} & 0_{r,n-r} \\ \circ_2 F_{21} \Xi_r^{1/2} & 0_{n-r} \end{bmatrix}) = \pm_1 (\Gamma_2)$$

Since $\pm_1(\Gamma_2) < 1 \Leftrightarrow \text{ª}_1(\Gamma_2^T \Gamma_2) < 1$, one finally has

$$1 > \text{ª}_1(\Gamma_2^T \Gamma_2) = \text{ª}_1 (\begin{bmatrix} \Xi_r^{1/2} F_{11}^T \Xi_r^{1/2} & \circ_2 \Xi_r^{1/2} F_{21}^T \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \begin{bmatrix} \Xi_r^{1/2} F_{11} \Xi_r^{1/2} & 0_{r,n-r} \\ \circ_2 F_{21} \Xi_r^{1/2} & 0_{n-r} \end{bmatrix})$$

$$= \text{ª}_1 (\begin{bmatrix} \Xi_r^{1/2} F_{11}^T \Xi_r F_{11} \Sigma_r^{1/2} + \circ_2^2 \Xi_r^{1/2} F_{21}^T F_{21} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}) \tag{3.7}$$

$$\Rightarrow \text{ª}_1(\Xi_r^{1/2} F_{11}^T \Xi_r F_{11} \Xi_r^{1/2}) = \pm_1^2(\Xi_r^{1/2} F_{11} \Xi_r^{1/2}) < 1$$

$$\Leftrightarrow \pm_1(\Xi_r^{1/2} F_{11} \Xi_r^{1/2}) < 1$$

which proves the necessity part of statement (ii) for Φ_2.

Suppose that $A \in \tilde{\boldsymbol{S}}$, i.e.

$$\pm_1(\Xi_r^{1/2} F_{11} \Xi_r^{1/2}) < 1 \Rightarrow \Psi_1 = \Xi_r^{1/2} F_{11} \Xi_r F_{11}^T \Xi_r^{1/2} < I_r$$

$$\pm_1(\Xi_r^{1/2} F_{11} \Xi_r^{1/2}) < 1 \Rightarrow \Psi_2 = \Xi_r^{1/2} F_{11}^T \Xi_r F_{11} \Xi_r^{1/2} < I_r$$

and consider (3.6) and (3.7). Obviously, there always exist some positive scalars $\circ_1$, $\circ_2$, such that

$$I_r > \Psi_1 + \circ_1^2 \Xi_r^{1/2} F_{12} F_{12}^T \Xi_r^{1/2} \tag{3.8}$$

$$I_r > \Psi_2 + \circ_2^2 \Xi_r^{1/2} F_{21}^T F_{21} \Xi_r^{1/2} \tag{3.9}$$

These inequalities guarantee that

$$\pm_1(\Phi_1^{-1/2} A \Phi_1^{1/2}) < 1, \quad \pm_1(\Phi_2^{1/2} A \Phi_2^{-1/2}) < 1$$

Therefore, Φ_1^{-1}, Φ_2 for γ_1, γ_2 satisfying (3.8) and (3.9), respectively, are Lyapunov matrices for *A*.

Lemma 3.1 The following statements hold:

(i) $A \in \boldsymbol{S}^{\prime}$, only if Φ_1^{-1}, Φ_2 in (3.4), (3.5) are Lyapunov matrices for some γ_1, γ_2 satisfying (3.8) and (3.9)

(ii) $A \in \tilde{\boldsymbol{S}}$ if and only if $\lambda_1(R_1 R_2) < 1$ where

$$R_1 = (AA^T)^{1/2}, \quad R_2 = (A^T A)^{1/2} \tag{3.10}$$

(iii) Let $A \in \tilde{\boldsymbol{S}}$. If the positive scalars γ_1, γ_2 are chosen to satisfy the inequalities

$$R_2^{1/2} R_1 R_2^{1/2} + \gamma_1^2 R_2 < I \Leftrightarrow \lambda_1[R_2(R_1 + \gamma_1^2 I)] < 1 \tag{3.11}$$

$$R_1^{1/2} R_2 R_1^{1/2} + \gamma_2^2 R_1 < I \Leftrightarrow \lambda_1[R_1(R_2 + \gamma_2^2 I)] < 1 \tag{3.12}$$

then, Φ_1^{-1} and Φ_2 are Lyapunov matrices for A.

(iv)If *A* is a nonsingular matrix belonging to the set $\boldsymbol{S}^{\prime}$, then R_1^{-1}, R_2 in (3.10) are Lyapunov matrices for it.

Proof. Statement (i) follows from Theorem 3.1.

(ii) Having in mind (3. 1)-(3.3) and (3.10) consider the matrix

$$R_1^{1/2} R_2 R_1^{1/2} = U\Sigma^{1/2}U^T V\Sigma V^T U\Sigma^{1/2}U^T = U\Sigma^{1/2}F^T \Sigma F \Sigma^{1/2} U^T$$

$$\Rightarrow \lambda_1(R_1 R_2) = \sigma_1^2(\Sigma^{1/2} F \Sigma^{1/2})$$

$$= \sigma_1^2\left(\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \right)$$

$$= \sigma_1^2(\Xi_r^{1/2} F_{11} \Xi_r^{1/2})$$

Therefore, if *A* belongs to the set $\tilde{\boldsymbol{S}}$, it follows that $\sigma_1^2(\Xi_r^{1/2} F_{11} \Xi_r^{1/2}) = \lambda_1(R_1 R_2) < 1$ and vice versa

(iii) Suppose that for some positive scalar γ_1 one has:

$$I > R_2^{1/2} R_1 R_2^{1/2} + \gamma_1^2 R_2 = V\Sigma^{1/2}V^T U\Sigma U^T V\Sigma^{1/2}V^T + \gamma_1^2 V\Sigma V^T$$

$$\Leftrightarrow I > \Sigma^{1/2}V^T U\Sigma U^T V\Sigma^{1/2} + \gamma_1^2 \Sigma$$

$$= \Sigma^{1/2} F \Sigma F^T \Sigma^{1/2} + \gamma_1^2 \Sigma$$

$$= \begin{bmatrix} \Xi_r^{1/2} F_{11} \Xi_r F_{11}^T \Sigma_r^{1/2} + \circ_1^2 \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}$$

$$\geq \begin{bmatrix} \Xi_r^{1/2} F_{11} \Xi_r F_{11}^T \Sigma_r^{1/2} + \circ_1^2 \Xi_r^{1/2} F_{12} F_{12}^T \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}$$

$$= \begin{bmatrix} \Xi_r^{1/2} F_{11} \Xi_r^{1/2} & \circ_1 \Xi_r^{1/2} F_{12} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \begin{bmatrix} \Xi_r^{1/2} F_{11}^T \Xi_r^{1/2} & 0_{r,n-r} \\ \circ_1 F_{12}^T \Xi_r^{1/2} & 0_{n-r} \end{bmatrix}$$

$$\Leftrightarrow 1 > \pm_1 \left(\begin{bmatrix} \Xi_r^{1/2} F_{11} \Xi_r^{1/2} & \circ_1 \Xi_r^{1/2} F_{12} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \right)$$

$$= \pm_1 \left(\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1 I_{n-r} \end{bmatrix} \right)$$

$$= \pm_1 \left(\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} F \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1 I_{n-r} \end{bmatrix} \right)$$

$$= \pm_1 \left(\begin{bmatrix} \Xi_r^{-1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1^{-1} I_{n-r} \end{bmatrix} \begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} F \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1 I_{n-r} \end{bmatrix} \right)$$

$$= \pm_1 \left(U \begin{bmatrix} \Xi_r^{-1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1^{-1} I_{n-r} \end{bmatrix} U^T U \begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} V^T U \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & \circ_1 I_{n-r} \end{bmatrix} U^T \right)$$

$$= \pm_1 (\Phi_1^{-1/2} A \Phi_1^{1/2})$$

$$\Leftrightarrow A^T \Phi_1^{-1} A - \Phi_1^{-1} < 0$$

It follows that Φ_1^{-1} is a Lyapunov matrix for *A*, in this case. Now, let

$$I > R_1^{1/2} R_2 R_1^{1/2} + \circ_2^2 R_1 = U \Sigma^{1/2} U^T V \Sigma V^T U \Sigma^{1/2} U^T + \circ_2^2 U \Sigma U^T$$

$$\Leftrightarrow I > \Sigma^{1/2} F^T \Sigma F \Sigma^{1/2} + \circ_2^2 \Sigma$$

$$= \begin{bmatrix} \Xi_r^{1/2} F_{11}^T \Xi_r F_{11} \Sigma_r^{1/2} + \circ_2^2 \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}$$

$$\geq \begin{bmatrix} \Xi_r^{1/2} F_{11}^T \Xi_r F_{11} \Sigma_r^{1/2} + \circ_2^2 \Xi_r^{1/2} F_{21}^T F_{21} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}$$

$$= \begin{bmatrix} \Xi_r^{1/2} F_{11}^T \Xi_r^{1/2} & {}^{\circ}{}_2 \Xi_r^{1/2} F_{21}^T \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \begin{bmatrix} \Xi_r^{1/2} F_{11} \Xi_r^{1/2} & 0_{r,n-r} \\ {}^{\circ}{}_2 F_{21} \Xi_r^{1/2} & 0_{n-r} \end{bmatrix}$$

$$\Leftrightarrow 1 > \pm_1 (\begin{bmatrix} \Xi_r^{1/2} F_{11} \Xi_r^{1/2} & 0_{r,n-r} \\ {}^{\circ}{}_2 F_{21} \Xi_r^{1/2} & 0_{n-r} \end{bmatrix})$$

$$= \pm_1 (\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & {}^{\circ}{}_2 I_{n-r} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix})$$

$$= \pm_1 (\begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & {}^{\circ}{}_2 I_{n-r} \end{bmatrix} F \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix})$$

$$= \pm_1 (V \begin{bmatrix} \Xi_r^{1/2} & 0_{r,n-r} \\ 0_{n-r,r} & {}^{\circ}{}_2 I_{n-r} \end{bmatrix} V^T U \begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} V^T V \begin{bmatrix} \Xi_r^{-1/2} & 0_{r,n-r} \\ 0_{n-r,r} & {}^{\circ}{}_2^{-1} I_{n-r} \end{bmatrix} V^T)$$

$$= \pm_1 (\Phi_2^{1/2} A \Phi_2^{-1/2})$$

$$\Leftrightarrow A^T \Phi_2 A - \Phi_2 < 0$$

which proves that Φ_2 is a Lyapunov matrix for *A*, in this case.

(iv) If $A \in \boldsymbol{S}^{\boldsymbol{1}}$, i.e. $AA^T = R_1^2 < I \Leftrightarrow A^T A = R_2^2 < I$, and in addition *A* is a nonsingular matrix, then $r = rank(A) = n$ and

$$\Sigma \equiv \Sigma_n ,\ \Phi_1 \equiv R_1 ,\ \Phi_2 \equiv R_2 \Leftrightarrow R_1 < I \Leftrightarrow R_2 < I$$

Consider the following inequalities:

$$\begin{aligned} A^T R_1^{-1} A - R_1^{-1} &= (V \Sigma U^T)(U \Sigma^{-1} U^T)(U \Sigma V^T) - R_1^{-1} \\ &= (V \Sigma V^T) - R_1^{-1} = R_2 - R_1^{-1} \\ &= R_2 - R_1^{-1} \\ &< I - R_1^{-1} \\ &< 0 \end{aligned}$$

and

$$\begin{aligned} A^T R_2 A - R_2 &< A^T A - R_2 \\ &= R_2^2 - R_2 \\ &= R_2^{1/2} (R_2 - I) R_2^{1/2} \\ &< 0 \end{aligned}$$

This proves the statement.

Remark 3.1 Theorem 3.1 and Lemma 3.1 can be viewed upon as corresponding to Theorem 2.1 for the discrete-time case. They show how the conservative set $\boldsymbol{S}^{1}$ can be extended to the set $\tilde{\boldsymbol{S}}$. Also, a natural accordance between the sets $\boldsymbol{H}^{-}$, $\tilde{\boldsymbol{H}}$ (continuous-time) and $\boldsymbol{S}^{1}$, $\tilde{\boldsymbol{S}}$ (discrete-time) exists, i.e. $\boldsymbol{H}^{-}$ corresponds to $\boldsymbol{S}^{1}$, $\tilde{\boldsymbol{H}}$ corresponds to $\tilde{\boldsymbol{S}}$, and $\boldsymbol{H}^{-} \subseteq \tilde{\boldsymbol{H}}$, $\boldsymbol{S}^{1} \subseteq \tilde{\boldsymbol{S}}$. The extension of the matrix set for which computable upper bounds for the DALE solution exist is based on the fact that if the coefficient matrix *A* in (1.9) belongs to the set $\tilde{\boldsymbol{S}}$, then easily computable Lyapunov matrices exist for it.

Example 3.1 Consider the stable matrix

$$A = \begin{bmatrix} 0 & ® & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}, \quad ª_{1,2,3}(A) = 0, \quad ±_1(A) = \max(|®|, 0.5)$$

If $|®| \geq 1$, then $±_1(A) \geq 1$ and none of the upper bounds (1.39), (1.42) is computable. The application of the upper bounds (1.65)-(1.68) requires an additional computational procedure (see (1.63)). The next table illustrates how the parameter ® influences the singular value of *A* and the eigenvalue of the matrix product $R_1 R_2$.

®	$±_1(A)$	$ª_1(R_1 R_2)$
4	4	1
3	3	0.75
2	2	0.5
1	1	0.25
0.25	0.25	0.125
0	0.25	0

Table 3.1 Dependence of $±_1(A)$ and $ª_1(R_1 R_2)$ on ®

It follows that $A \in \boldsymbol{S}^{1}$ is satisfied for $|®| < 1$, while $A \in \tilde{\boldsymbol{S}}$, for $|®| < 4$.

3.2 BOUNDS FOR THE DALE SOLUTION

3.2.1 TRACE BOUNDS

The significance of the singular value decomposition approach in getting less restrictive conditions for the validity of various scalar and matrix upper bounds will be illustrated. Before that, consider the following result.

Let X be a Lyapunov matrix for A. Then, one has:

$$\begin{aligned} A^T XA - X < 0 \Leftrightarrow 1 &> \pm_1(X^{1/2}AX^{-1/2}) \\ &= \pm_1(X^{-1/2}A^T X^{1/2}) \\ &\Leftrightarrow AX^{-1}A^T - X^{-1} < 0 \end{aligned} \qquad (3.13)$$

Therefore, X is a Lyapunov matrix for A, if and only if X^{-1} is a Lyapunov matrix for A^T.

Remark 3.2 In what follows, it is assumed that under the condition $A \in \tilde{\boldsymbol{S}}$, the parameters $^\circ{}_1$, $^\circ{}_2$ are chosen to satisfy the exact conditions (3.8), (3.9), or the sufficient conditions (3.11), (3.12) , respectively, i.e. Φ_1, Φ_2 given in (3.4), (3.5) are Lyapunov matrices for the coefficient matrix A.

Theorem 3.2 Suppose that $A \in \tilde{\boldsymbol{S}}$. The trace of the DALE solution (1.9) can be bounded from above as follows:

$$tr(P) \le t_{U1} = \min\left(\frac{tr(\Phi_1 Q)}{\text{ª}_n(\Phi_1 - A\Phi_1 A^T)}, \frac{tr(\Phi_2^{-1}Q)}{\text{ª}_n(\Phi_2^{-1} - A\Phi_2^{-1}A^T)}\right) \qquad (3.14)$$

Proof Due to Theorem 3.1, $A \in \tilde{\boldsymbol{S}}$ if and only if Φ_1^{-1}, Φ_2 are Lyapunov matrices for the coefficient matrix A for some positive scalars $^\circ{}_1$, $^\circ{}_2$. By making use of (3.13), one gets

$$A\Phi_1 A^T - \Phi_1 < 0$$

$$A\Phi_2^{-1}A^T - \Phi_2^{-1} < 0$$

Now, consider the DALE pre-multiplied by Φ_1 , i.e.

$$\Phi_1 A^T PA - \Phi_1 P = -\Phi_1 Q$$

and by Φ_2^{-1} :

$$\Phi_2^{-1}A^T PA - \Phi_2^{-1}P = -\Phi_2^{-1}Q$$

Application of the trace operator to both sides results in:

$$tr(\Phi_1 Q) = tr[(\Phi_1 - A\Phi_1 A^T)P] \tag{3.15}$$
$$\geq \lambda_n(\Phi_1 - A\Phi_1 A^T)tr(P)$$

which proves the first bound in (3.14), and

$$tr(\Phi_2^{-1} Q) = tr[(\Phi_2^{-1} - A\Phi_2^{-1} A^T)P] \tag{3.16}$$
$$\geq \lambda_n(\Phi_2^{-1} - A\Phi_2^{-1} A^T)tr(P)$$

and the second bound in (3.14) follows.

Theorem 3.3 Let $A \in \tilde{\boldsymbol{S}}$. The trace of the DALE solution (1.9) has the following upper bound:

$$tr(P) \leq t_{U2} = \min\left(\frac{1}{\lambda_n(\Phi_1)}\sum_{i=1}^{n}\frac{\lambda_i(Q\Phi_1)}{[1-\sigma_i^2(\Phi_1^{-1/2}A\Phi_1^{1/2}]},\ \lambda_1(\Phi_2)\sum_{i=1}^{n}\frac{\lambda_i(Q\Phi_2^{-1})}{[1-\sigma_i^2(\Phi_2^{1/2}A\Phi_2^{-1/2})]}\right) \tag{3.17}$$

where Φ_1^{-1}, Φ_2 in (3.4), (3.5), are Lyapunov matrices for *A*.

Proof $A \in \tilde{\boldsymbol{S}}$, if and only if Φ_1^{-1}, Φ_2 are Lyapunov matrices for *A*. Consider the DALE (1.9) pre- and post-multiplied by $\Phi_1^{1/2}$

$$-\Phi_1^{1/2}Q\Phi_1^{1/2} = \Phi_1^{1/2}A^T PA\Phi_1^{1/2} - \Phi_1^{1/2}P\Phi_1^{1/2}$$
$$= (\Phi_1^{1/2}A^T\Phi_1^{-1/2})(\Phi_1^{1/2}P\Phi_1^{1/2})(\Phi_1^{-1/2}A\Phi_1^{1/2}) - \Phi_1^{1/2}P\Phi_1^{1/2}$$
$$= \tilde{A}_1^T\tilde{P}_1\tilde{A}_1 - \tilde{P}_1 \tag{3.18}$$

and by $\Phi_2^{-1/2}$

$$-\Phi_2^{-1/2}Q\Phi_2^{-1/2} = \Phi_2^{-1/2}A^T PA\Phi_2^{-1/2} - \Phi_2^{-1/2}P\Phi_2^{-1/2}$$
$$= (\Phi_2^{-1/2}A^T\Phi_1^{1/2})(\Phi_2^{-1/2}P\Phi_2^{-1/2})(\Phi_2^{1/2}A\Phi_2^{-1/2}) - \Phi_2^{-1/2}P\Phi_2^{-1/2}$$
$$= \tilde{A}_2^T\tilde{P}_2\tilde{A}_2 - \tilde{P}_2 \tag{3.19}$$

where $\tilde{A}_1 = \Phi_1^{-1/2}A\Phi_1^{1/2}$ and $\tilde{A}_2 = \Phi_2^{1/2}A\Phi_2^{-1/2}$. Note, that $\sigma_1(\tilde{A}_1) < 1$, and $\sigma_1(\tilde{A}_2) < 1$, which makes possible to apply the estimates in (1.65), (1.67) for the modified DALEs (3.18) and (3.19), which leads to the bound in (3.17).

Theorem 3.4 The trace of the DALE solution has the following lower bounds:

$$tr(P) \geq t_{L1} = \max(t_1, t_2, t_3) \tag{3.20}$$

where

$$t_1 = \frac{tr(\Phi_1 Q)}{\lambda_1(\Phi_1 - A\Phi_1 A^T)}, \quad t_2 = \frac{tr(\Phi_2^{-1} Q)}{\lambda_1(\Phi_2^{-1} - A\Phi_2^{-1} A^T)}, \quad t_3 = tr\{\sum_{i=0}^{k}[A^i (A^i)^T]Q\}, \, k = 0,1,...$$

Proof The left hand-sides in (3.15) and (3.16) can be estimated as follows:

$$tr(\Phi_1 Q) = tr[(\Phi_1 - A\Phi_1 A^T)P]$$
$$\leq \lambda_1(\Phi_1 - A\Phi_1 A^T)tr(P)$$

and

$$tr(\Phi_2^{-1} Q) = tr[(\Phi_2^{-1} - A\Phi_2^{-1} A^T)P]$$
$$\leq \lambda_1(\Phi_2^{-1} - A\Phi_2^{-1} A^T)tr(P)$$

and the first two bounds in (3.20) are proved. Having in mind (1.10), the third one follows.

3.2.2 UPPER EIGENVALUE BOUNDS

The singular value decomposition approach can be applied to derive bounds for the maximal eigenvalue of the DALE solution matrix under relaxed validity constraints.

Theorem 3.5 The maximal eigenvalue of P in (1.9) has the following upper bound:

$$\lambda_1(P) \leq e_{U1} = \min(u_1, u_2, u_2) \tag{3.21}$$

$$u_1 = \lambda_1[Q(I - A^T A)^{-1}], \quad A \in \boldsymbol{S}^{1}$$

$$u_2 = \frac{\lambda_1[Q(\Phi_1^{-1} - A^T \Phi_1^{-1} A]}{\lambda_n(\Phi_1)}, \quad A \in \tilde{\boldsymbol{S}}$$

$$u_3 = \lambda_1[Q(\Phi_2 - A_2^T \Phi_2 A_2)^{-1}]\lambda_1(\Phi_2), \quad A \in \tilde{\boldsymbol{S}}$$

Proof Suppose that $\pm_1(A) < 1$, let x be an eigenvector corresponding to the maximal eigenvalue of P and consider the DALE (1.9) pre- and post-multiplied by the vectors x^T, x, respectively, i.e.

$$x^T P x = \lambda_1(P) x^T x = x^T A^T P A x + x^T Q x$$
$$\leq \lambda_1(P) x^T A^T A x + x^T Q x$$
$$\Rightarrow x^T (I - A^T A) x \lambda_1(P) \leq x^T Q x$$
$$\Rightarrow \lambda_1(P) \leq \frac{x^T Q x}{x^T (I - A^T A) x}$$

$$= \frac{y^T S^{-1/2} Q S^{-1/2} y}{y^T y},\ S = (I - A^T A), y = S^{1/2} x$$

$$\leq \lambda_1(QS^{-1}),\ \forall x \neq 0$$

This proves the first bound in (3.21).

Let $A \in \tilde{\boldsymbol{S}}$, i.e. Φ_1^{-1}, Φ_2 are Lyapunov matrices for *A*. Having in mind, that $\sigma_1(\tilde{A}_1) < 1$, and $\sigma_1(\tilde{A}_2) < 1$, application of bound u_1 for (3.18) and (3.19) results in the following bounds for their solution matrices $\tilde{P}_1$, $\tilde{P}_2$:

$$\begin{aligned}
\lambda_1(\tilde{P}_1) &\leq \lambda_1[\Phi_1^{1/2} Q \Phi_1^{1/2} (I - \tilde{A}_1^T \tilde{A}_1)^{-1}] \\
&= \lambda_1[Q \Phi_1^{1/2} (I - \tilde{A}_1^T \tilde{A}_1)^{-1} \Phi_1^{1/2}] \\
&= \lambda_1\{Q[\Phi_1^{-1/2} (I - \tilde{A}_1^T \tilde{A}_1) \Phi_1^{-1/2}]^{-1}\} \\
&= \lambda_1[Q(\Phi_1^{-1} - A^T \Phi_1^{-1} A)^{-1}]
\end{aligned}$$

and

$$\begin{aligned}
\lambda_1(\tilde{P}_2) &\leq \lambda_1[\Phi_2^{-1/2} Q \Phi_2^{-1/2} (I - \tilde{A}_2^T \tilde{A}_2)^{-1}] \\
&= \lambda_1[Q \Phi_2^{-1/2} (I - \tilde{A}_2^T \tilde{A}_2)^{-1} \Phi_2^{-1/2}] \\
&= \lambda_1\{Q[\Phi_2^{1/2} (I - \tilde{A}_2^T \tilde{A}_2) \Phi_2^{1/2}]^{-1}\} \\
&= \lambda_1[Q(\Phi_2 - A_2^T \Phi_2 A_2)^{-1}]
\end{aligned}$$

The maximal eigenvalues of $\tilde{P}_1$ and $\tilde{P}_2$ can be estimated from below as:

$$\lambda_1(\tilde{P}_1) = \lambda_1(\Phi_1 P) \geq \lambda_n(\Phi_1)\lambda_1(P), \quad \lambda_1(\tilde{P}_2) = \lambda_1(\Phi_2^{-1} P) \geq \lambda_n(\Phi_2^{-1})\lambda_1(P) = \frac{\lambda_1(P)}{\lambda_1(\Phi_2)}$$

and the respective bounds in (3.21) are proved.

Remark 3.3 Theorem 3.4 can be generalized for arbitrary positive definite matrix *T*, such that $\sigma_1(\tilde{A}) < 1$, where $\tilde{A} = T^{-1} A T$, i.e. T^{-2} is a Lyapunov matrix for *A*. Consider the modified DALE (1.64) and the applied for *A* and *Q* replaced with $\tilde{A}$ and $\tilde{Q} = TQT$, bound u_1 in (3.21), which results in the following estimate for the maximal eigenvalue of $\tilde{P} = TPT$ in (1.64) and *P* in (1.9):

$$\lambda_1(T^2 P) \leq \lambda_1[\tilde{Q}(I - \tilde{A}^T \tilde{A})^{-1}] \Rightarrow \lambda_1(P) \leq \frac{\lambda_1[\tilde{Q}(I - \tilde{A}^T \tilde{A})^{-1}]}{\lambda_n(T^2)} \qquad (3.22)$$

The upper bound (3.22) for the maximal eigenvalue of the DALE solution P is tighter than the bound in (1.65), since

$$\lambda_1[\tilde{Q}(I-\tilde{A}^T\tilde{A})^{-1}] \le \lambda_1(\tilde{Q})\lambda_1[(I-\tilde{A}^T\tilde{A})^{-1}] = \frac{\lambda_1(QT^2)}{\lambda_n(I-\tilde{A}^T\tilde{A})} = \frac{\lambda_1(QT^2)}{1-\sigma_1^2(\tilde{A})}$$

If $\sigma_1(A)<1$, then for $T = I$, one has $\tilde{A}\equiv A$, and the bound u_1 in (3.21) is obtained. This estimate is tighter than the similar bound in (1.39).

Corollary 3.1 Let A be a nonsingular matrix belonging to the set $\tilde{\boldsymbol{S}}$. The trace and maximal eigenvalue of the DALE solution can be estimated as follows:

$$tr(P) \le t_{U3} = \min\left(\frac{tr(R_1Q)}{\lambda_n(R_1 - AR_1A^T)}, \frac{tr(R_2^{-1}Q)}{\lambda_n(R_2^{-1} - AR_2^{-1}A)}\right) \tag{3.23}$$

$$\lambda_1(P) \le e_{U2} = \min\left(\frac{\lambda_1[Q(R_1^{-1} - A^TR_1^{-1}A]}{\sigma_n(A)},\ \sigma_1(A)\lambda_1[Q(R_2 - A^TR_2A)^{-1}]\right) \tag{3.24}$$

where $R_1 = (AA^T)^{1/2}$ and $R_2 = (A^TA)^{1/2}$.

Proof Having in mind (3.1), (3.2), (3.4) and (3.5), if A is a nonsingular matrix, one has

$$r = rank(A) = n, \quad \Phi_1 \equiv R_1, \quad \Phi_2 \equiv R_2$$

In accordance with Theorem 3.1, Lemma 3.1 and (3.11), it follows that R_1^{-1}, R_2 are Lyapunov matrices for A, or, equivalently, R_1, R_2^{-1} are Lyapunov matrices for A^T. Bounds (3.23) and (3.24) follow from the respective estimates (3.14) and (3.21), when the non-singularity of A is taken into account.

3.2.3 MATRIX BOUNDS

Theorem 3.6 Suppose that $A \in \boldsymbol{S^1}$. For arbitrary $p = 1,2,..., a, q = 0,1,2,...$, the following statements hold:

(i) the DALE solution matrix has the following upper bound:

$$P \le P_{U,p} = Q + \lambda_1(Q)[\sum_{i=1}^{p-1}(A^i)^TA^i + \frac{1}{1-\sigma_1^2(A)}(A^p)^TA^p] - S(a,q) \tag{3.25}$$

$$S(a,q) = A(a) + Q(q) \tag{3.26}$$

$$A(a) = \text{ª}_1(Q)\sum_{i=1}^{a}\sum_{s=0}^{i-1}\pm_1^{2(i-1-s)}(A^{s+p})^T\Delta_A A^{s+p},\ Q(q) = \sum_{i=1}^{q}(A^i)^T\Delta_Q A^i \qquad (3.27)$$

$$\Delta_A = \pm_1^2(A)I - A^T A, \quad \Delta_Q = \text{ª}_1(Q)I - Q \qquad (3.28)$$

(ii) $$P_{U,p} \le P_{U,p-1},\ \forall p$$

Proof Having in mind (3.28), the solution of the DALE (1.10) can be rewritten as follows:

$$P = \sum_{i=0}^{\infty}(A^i)^T Q A^i$$

$$= Q + \sum_{i=1}^{\infty}(A^i)^T Q A^i$$

$$= Q + \text{ª}_1(Q)\sum_{i=1}^{\infty}(A^i)^T A^i - \sum_{i=1}^{\infty}(A^i)^T\Delta_Q A^i$$

$$= Q + \text{ª}_1(Q)\sum_{i=1}^{\infty}(A^i)^T A^i - Q(\infty)$$

$$= Q + \text{ª}_1(Q)\{\sum_{i=1}^{p-1}[(A^i)^T A^i] + (A^P)^T[I + \sum_{i=1}^{\infty}(A^i)^T A^i]A^p\} - Q(\infty) \qquad (3.29)$$

Consider the matrix

$$(A^i)^T A^i = (A^{i-1})^T A^T A(A^{i-1})$$

$$= \pm_1^2(A)(A^{i-1})^T(A^{i-1}) - (A^{i-1})^T\Delta_A A^{i-1}$$

$$= \pm_1^4(A)(A^{i-2})^T(A^{i-2}) - \pm_1^2(A)(A^{i-2})^T\Delta_A A^{i-2} - (A^{i-1})^T\Delta_A A^{i-1}$$

$$\ldots$$

$$= \pm_1^{2i}(A)I - \sum_{s=0}^{i-1}\pm_1^{2(i-1-s)}(A^s)^T\Delta_A A^s \qquad (3.30)$$

Substitution of (3.30) into (3.29) results in:

$$P = Q + \text{ª}_1(Q)\{\sum_{i=1}^{p-1}[(A^i)^T A^i] + [1 + \sum_{i=1}^{\infty}\pm_1^{2i}(A)](A^p)^T(A^p)^T - \sum_{i=1}^{\infty}\sum_{s=0}^{i-1}\pm_1^{2(i-s-1)}(A^{s+p})^T\Delta_A A^{s+p}\} - Q(\infty)$$

$$= Q + \text{ª}_1(Q)\{\sum_{i=1}^{p-1}[(A^i)^T A^i] + [1 + \sum_{i=1}^{\infty}\pm_1^{2i}(A)](A^p)^T(A^p)^T\} - A(\infty) - Q(\infty)$$

$$= Q + \text{ª}_1(Q)\{\sum_{i=1}^{p-1}[(A^i)^T A^i] + \sum_{i=0}^{\infty}\pm_1^{2i}(A)[(A^p)^T(A^p)^T]\} - S(\infty,\infty)$$

It is well known, that for any given scalar $\text{¥}, \text{¥}^2 < 1$, one has:

$$\sum_{i=0}^{\infty} ¥^{2i} = 1 + ¥^{2} + ¥^{4} + ¥^{6} + ... = \frac{1}{1-¥^{2}}$$

Therefore, the solution of the CALE is given by:

$$P = Q + ª_1(Q)\{\sum_{i=1}^{p-1}[(A^i)^T A^i] + \frac{1}{1-±_1^2(A)}(A^p)^T A^p\} - S(\infty,\infty)$$

The matrices in (3.28) are positive (semi)-definite, which makes possible to estimate the sum $S(\infty,\infty)$ in (3.26), (3.27), as follows:

$$S(\infty,\infty) = A(\infty) + Q(\infty) \geq S(a,q) = A(a) + Q(q)$$

for arbitrary finite integers a = 0, 1,.., and q = 0, 1, …. This proves statement (i).

Consider the difference matrix:

$$P_{U,p} = P_{U,p-1} = ª_1(Q)\{(A^{p-1})^T A^{p-1} + \frac{1}{1-±_1^2(A)}[(A^p)^T A^p - (A^{p-1})^T A^{p-1}]\}$$

$$= ª_1(Q)(A^{p-1})^T[I + \frac{1}{1-±_1^2(A)}(A^T A - I)]A^{p-1}$$

$$= \frac{ª_1(Q)}{1-±_1^2(A)}[A^T A - ±_1^2(A) I]$$

$$= -\frac{ª_1(Q)}{1-±_1^2(A)}\Delta_A$$

$$\leq 0$$

$$\Rightarrow P_{U,p} - P_{U,p-1} \leq 0$$

This proves the second statement of the Theorem.

Theorem 3.7 Suppose that $±_1(\tilde{A}) < 1$, $\tilde{A} = T^{-1}AT$, for some positive definite matrix T. The solution of the DALE (1.9) has the following upper matrix bound:

$$P \leq \overline{P}_{U,p} = Q + ª_1(\tilde{Q})[\sum_{i=1}^{p-1}(A^i)^T T^{-2} A^i + \frac{1}{1-±_1^2(\tilde{A})}(A^p)^T T^{-2} A^p] - \overline{S}(a,q) \qquad (3.31)$$

for all $p = 1,2,...,\ a,q = 0,1,2,...$, where $\tilde{Q} = TQT$ and

$$\overline{S}(a,q) = \overline{A}(a) + \overline{Q}(q) \qquad (3.32)$$

$$\overline{A}(a) = ª_1(\tilde{Q})\sum_{i=1}^{a}\sum_{s=0}^{i-1} ±_1^{2(i-1-s)}(A^{s+1})^T \overline{\Delta}_A A^{s+1}, \quad \overline{\Delta}_A = ±_1^2(\tilde{A})T^{-2} - A^T T^{-2} A \qquad (3.33)$$

$$\overline{Q}(q) = [\sum_{i=1}^{q}(A^i)^T \overline{\Delta}_Q A^i], \quad \overline{\Delta}_Q = \text{ª}_1(\tilde{Q})T^{-2} - Q \tag{3.34}$$

Proof. Consider the modified DALE (1.64). If $\pm_1(\tilde{A}) < 1$, then application of the bound (3.25) for the solution $\tilde{P} = TPT$ in (1.64) results in the estimate

$$\tilde{P} \leq \tilde{P}_{U,p} = \tilde{Q} + \text{ª}_1(\tilde{Q})[\sum_{i=1}^{p-1}(\tilde{A}^i)^T \tilde{A}^i + \frac{1}{1-\pm_1^2(\tilde{A})}(\tilde{A}^p)^T \tilde{A}^p] - \tilde{S}(a,q)$$

where

$$\tilde{S}(a,q) = \tilde{A}(a) + \tilde{Q}(q)$$

$$\tilde{A}(a) = \text{ª}_1(\tilde{Q})\sum_{i=1}^{a}\sum_{s=0}^{i-1}\pm_1^{2(i-1-s)}(\tilde{A}^{s+1})^T \tilde{\Delta}_A \tilde{A}^{s+1}, \; \tilde{Q}(q) = \sum_{i=1}^{q}(\tilde{A}^i)^T \tilde{\Delta}_Q \tilde{A}^i$$

$$\Delta_{\tilde{A}} = \pm_1^2(\tilde{A})I - \tilde{A}^T \tilde{A}, \quad \Delta_{\tilde{Q}} = \text{ª}_1(\tilde{Q})I - \tilde{Q}$$

Since, $\tilde{A}^t = T^{-1}A^t T$, $\forall t = 0,1,..$, we obtain the transformed matrices

$$\tilde{A}(a) = \text{ª}_1(\tilde{Q})T[\sum_{i=1}^{a}\sum_{s=0}^{i-1}\pm_1^{2(i-1-s)}(A^{s+1})^T \overline{\Delta}_A A^{s+1}]T$$

$$\tilde{Q}(q) = T[\sum_{i=1}^{q}(A^i)^T \overline{\Delta}_Q A^i]T$$

where $\overline{\Delta}_A, \overline{\Delta}_Q$ are given in (3.33), (3.34), respectively. Then,

$$\tilde{P} = TPT \leq \tilde{P}_{U,p} = TQT + \text{ª}_1(\tilde{Q})T[\sum_{i=1}^{p-1}(A^i)^T T^{-2} A^i + \frac{1}{1-\pm_1^2(\tilde{A})}(A^p)^T T^{-2} A^p]T - \tilde{S}(a,q)$$

Finally, the DALE solution matrix P is estimated as:

$$P \leq \overline{P}_{U,p} = Q + \text{ª}_1(\tilde{Q})[\sum_{i=1}^{p-1}(A^i)^T T^{-2} A^i + \frac{1}{1-\pm_1^2(\tilde{A})}(A^p)^T T^{-2} A^p] - \overline{S}(a,q)$$

where the positive (semi)-definite matrix $\overline{S}(a,q)$ is given in (3.32)-(3.34). This proves the upper matrix bound (3.31).

Remark 3.4 Consider the upper matrix bounds P_U (1.68) and $P_{U,p}$ (3.31). For p =1, and a = q = 0, one gets $P_U = P_{U,1}$. In accordance with statement (ii) in Theorem 3.6, it follows that $P_U \geq P_{U,p}$, $\forall p \geq 2, \forall a,q \geq 0$.

Corollary 3.2 Suppose that $A \in \tilde{S}$ and consider the bound in (3.31)-(3.34). The DALE solution matrix has the upper bounds

$$P \le P_{U1,p} = \overline{P}_{U,p}, \quad T^2 = \Phi_1 \tag{3.35}$$

$$P \le P_{U2,p} = \overline{P}_{U,p}, \quad T^2 = \Phi_2^{-1} \tag{3.36}$$

for arbitrary $p = 1,2,...,\ a, q = 0,1,2,...$, where the upper bound $\overline{P}_{U,p}$ is given in (3.31), Φ_1^{-1} (3.4) and Φ_2 (3.5) are Lyapunov matrices for A in accordance with statement (iii) in Lemma 3.1. Also,

$$P_{U1,p} \le P_{U1,p-1}, \qquad P_{U2,p} \le P_{U2,p-1}, \forall p \tag{3.37}$$

Proof. Under the supposition that A belongs to the set $\tilde{S}$, and having in mind Lemma 3.1, it follows that Φ_1^{-1}(3.4) and Φ_2(3.5) are Lyapunov matrices for A, i.e.

$$±_1(\tilde{A}) < 1, \quad T^{-1}AT, \quad T = \Phi_1^{1/2}, \quad T = \Phi_2^{-1/2}$$

The bounds (3.35), (3.36) and the matrix inequalities (3.37) follow from Theorem 3.7, when applied for $T = \Phi_1^{1/2}, T = \Phi_2^{-1/2}$

Theorem 3.8 [118] If $A \in \tilde{S}$, the following upper matrix bounds for the DALE solution hold:

$$P \le P_{U3} = «_{U1}\Phi_1^{-1}, \quad «_{U1} = ª_1[Q(\Phi_1^{-1} - A^T\Phi_1^{-1}A)^{-1}] \tag{3.38}$$

$$P \le P_{U4} = «_{U2}\Phi_2, \quad «_{U2} = ª_1[Q(\Phi_2 - A^T\Phi_2 A)^{-1}] \tag{3.39}$$

where the matrices Φ_1^{-1}(3.4) and Φ_2(3.5) are Lyapunov matrices for A in accordance with statement (iii) in Lemma 3.1

Proof. Under the supposition that $A \in \tilde{S}$, the matrices in (3.4) and (3.5) can be always chosen to satisfy the inequalities:

$$A^T\Phi_1^{-1}A - \Phi_1^{-1} < 0, \; A^T\Phi_2 A - \Phi_2 < 0$$

Consider the positive scalars $«_{U1}$ and $«_{U2}$ in (3.38) and (3.39), respectively. From their definition it follows, that

$$«_{U1}I \ge [\Phi_1^{-1} - A^T\Phi_1^{-1}A]^{-1/2}Q[\Phi_1^{-1} - A^T\Phi_1^{-1}A)]^{-1/2}$$

$$\Rightarrow «_{U1}[\Phi_1^{-1} - A^T\Phi_1^{-1}A] \ge Q$$

$$\Rightarrow 0 > Q + A^T(«_{U1}\Phi_1^{-1})A - («_{U1}\Phi_1^{-1})$$

$$= Q + A^T P_{U3} A - P_{U3}$$

and

$$«_{U2} I \geq [\Phi_2 - A^T\Phi_2 A]^{-1/2} Q [\Phi_2 - A^T\Phi_2 A)]^{-1/2}$$

$$\Rightarrow «_{U2}[\Phi_2 - A^T\Phi_2 A] \geq Q$$

$$\Rightarrow 0 > Q + A^T(«_{U2}\Phi_2)A - («_{U2}\Phi_2)$$

$$= Q + A^T P_{U4} A - P_{U4}$$

Consider the DALE (1.9) rewritten as:

$$A^T(P_{U3} - P)A - (P_{U3} - P) = Q + A^T P_{U3} A - P_{U3} \leq 0$$

$$A^T(P_{U4} - P)A - (P_{U4} - P) = Q + A^T P_{U4} A - P_{U4} \leq 0$$

From Theorem 1.8 it follows that

$$P_{U3} - P \geq 0, \quad P_{U4} - P \geq 0$$

This proves the bounds in (3.38) and (3.39).

Corollary 3.3 The maximal eigenvalue and the trace of the DALE solution have the following upper bounds:

$$ª_1(P) \leq e_{U3,p} = ª_1(P_{U,p}), \ \forall p = 1,2,..., \forall a, q = 0,1,2,... \quad \text{if } A \in \boldsymbol{S}^1 \qquad (3.40)$$

$$ª_1(P) \leq e_{U4,p} = \min\{ª_1(P_{U1,p}), ª(P_{U2,p})\}, \ \forall p = 1,2,..., \forall a, q = 0,1,2,... \quad \text{if } A \in \tilde{\boldsymbol{S}} \qquad (3.41)$$

$$tr(P) \leq t_{U4,p} = tr(P_{U,p}), \ \forall p = 1,2,..., \forall a, q = 0,1,2,... \quad \text{if } A \in \boldsymbol{S}^1 \qquad (3.42)$$

$$tr(P) \leq t_{U5,p} = \min(t_{U6,p}, t_{U7}) \qquad (3.43)$$

$$tr(P) \leq t_{U6,p} = \min\{tr(P_{U1,p}), tr(P_{U2,p})\}, \ \forall p = 1,2,..., \forall a, q = 0,1,2,... \quad \text{if } A \in \tilde{\boldsymbol{S}} \qquad (3.44)$$

$$tr(P) \leq t_{U7} = \min\{tr(P_{U3}), tr(P_{U4})\} \qquad \text{if } A \in \tilde{\boldsymbol{S}} \qquad (3.45)$$

where the upper matrix bounds $P_{U,p}$, $P_{U1,p}$, $P_{U2,p}$, P_{U3}, P_{U4}, for P in (1.9) are given in (3.25), (3.35), (3.36), (3.38), (3.39), respectively.

Due to (1.10), the DALE solution matrix can be estimated from below as:

$$P \geq P_{L,k} = \sum_{i=0}^{k} (A^k)^T Q A^k, \quad \forall k = 0,1,2...$$

If $Q = C^T C$ is a singular matrix, but (A, C) is an observable pair, then P is the unique positive definite solution of the DALE (see Theorem 1.4). This condition is equivalent to

$$0 < Q + A^T QA + (A^2)^T QA^2 + ... + (A^{n-1})^T QA^{n-1} = P_{L,n-1}$$

Therefore, the solution P can be always bounded from below by a strictly positive definite matrix $P_{L,k}$ with $k \leq n-1$, in this case. Also, using this lower bound, the minimal eigenvalue of P has the nontrivial lower estimate

$$\text{ª}_n(P) = e_{L1} = \text{ª}_n(P_{L,k}), \quad P_{L,k} > 0, \quad k \leq n-1 \tag{3.46}$$

The bound $P_{L,k}$ is actually used to get the lower trace bound t_3 in (3.20).

3.3 IMPROVEMENT OF BOUNDS

Under some condition there exists a procedure via which any upper matrix and scalar bound for the DALE solution can be improved in sense of tightness.

Theorem 3.9 Suppose that there exists a matrix P_U, satisfying the inequality

$$A^T P_U A - P_U + Q \leq 0 \tag{3.47}$$

Then, for any $i = 1, 2,...$ one has:

$$U_i \leq U_{i-1} \tag{3.48}$$

$$P \leq U_i = (A^i)^T P_U A^i + \sum_{j=0}^{i-1} (A^j)^T QA^j \tag{3.49}$$

Proof. Denote $\Delta_{i,i-1} = U_i - U_{i-1}, i = 1,2,.., U_0 \equiv P_U$, and consider the difference matrix

$$\begin{aligned}\Delta_{i,i-1} &= (A^i)^T P_U A^i + \sum_{j=0}^{i-1}(A^j)^T QA^j - (A^{i-1})^T P_U A^{i-1} - \sum_{j=0}^{i-2}(A^j)^T QA^j \\ &= (A^i)^T P_U A^i - (A^{i-1})^T P_U A^{i-1} + (A^{i-1})^T QA^{i-1} \\ &= (A^{i-1})^T [A^T P_U A - P_U + Q] A^{i-1} \\ &\leq 0\end{aligned}$$

This proves the inequalities in (3.48). Now, if the inequality (3.47) is satisfied, then the DALE is rewritten as

$$A^T (P_U - P)A - (P_U - P) = Q + A^T P_U A - P_U \leq 0$$

Having in mind, that A is a stable matrix, this is possible only if $P \leq P_U = U_0$. From the DALE (1.9) one gets:

$$
\begin{aligned}
& P = A^T P A + Q \\
& \Rightarrow P \leq A^T P_U A + Q = U_1 \\
& \Rightarrow P \leq A^T U_1 A + Q \\
& = (A^2)^T P_U A^2 + A^T Q A + Q = U_2 \\
& \ldots\ldots \\
& \Rightarrow P \leq U_i
\end{aligned}
$$

This proves that $U_i, i = 1, 2, ...$ is an upper matrix bound for the DALE solution.

Corollary 3.4 Let (3.47) holds for some symmetric matrix P_U. The maximal eigenvalue and the trace of the DALE solution have the bounds

$$a_1(P) \leq e_{U4,i} = a_1(U_i),\ i = 1, 2, ... \tag{3.50}$$

$$tr(P) \leq t_{U8,i} = tr(U_i),\ i = 1, 2, ... \tag{3.51}$$

with

$$e_{U4,i} \leq e_{U4,i-1},\ i = 1, 2, ... \tag{3.52}$$

$$t_{U8,i} \leq t_{U8,i-1}\ i = 1, 2, ... \tag{3.53}$$

Corollary 3.5 Let $A \in \tilde{\boldsymbol{S}}$. For any given positive integer *i*, the DALE solution matrix, its maximal eigenvalue and its trace have the following upper bounds:

$$P \leq U_{1i} = (A^i)^T P_{U3} A^i + \sum_{j=0}^{i-1} (A^j)^T Q A^j\ ,\ \ U_{1_i} \leq U_{1_{i-1}} \tag{3.54}$$

$$P \leq U_{2i} = (A^i)^T P_{U4} A^i + \sum_{j=0}^{i-1} (A^j)^T Q A^j\ ,\ \ U_{2_i} \leq U_{2_{i-1}} \tag{3.55}$$

$$a_1(P) \leq e_{U5,i} = \min\{a_1(U_{1_i}),\ a_1(U_{2_i})\},\ \ e_{U5,i} \leq e_{U5,i-1} \tag{3.56}$$

$$tr(P) \leq t_{U9,i} = \min\{tr(U_{1_i}),\ tr(U_{2_i})\},\ \ t_{U9,i} \leq t_{U9,i-1} \tag{3.57}$$

where matrices P_{U3}, P_{U4} are given in (3.38), (3.39), respectively.

Proof If $A \in \tilde{\boldsymbol{S}}$, the upper matrix bounds (3.38) and (3.39) for the DALE solution satisfy the inequality (3.47). In other words, P_{U3}, P_{U4} meet the supposition made in Theorem 3.8 and the inequalities (3.49) hold for $P_U = P_{U3}, P_{U4}$, which proves the matrix bounds (3.54) and (3.55). Then, the scalar bounds (3.56) and (3.57) follow.

It has been shown how upper matrix bounds can be improved in sense of tightness. As a consequence, such an improvement can be achieved for the maximal eigenvalue and the trace bounds. Moreover, all upper bounds based on the condition $A \in \tilde{\boldsymbol{S}}$ are less conservative, with respect to validity requirements, than the ones requiring $A \in \boldsymbol{S}^{\boldsymbol{1}}$.

The next results illustrate another approach for improvement of upper trace bounds for the DALE solution matrix.

Theorem 3.10 Let P_U be a symmetric matrix satisfying the inequality (3.47) and some nonnegative scalars $l_i \leq \text{ª}_i(P), i = 2,..,n$ exist. The trace of the DALE solution has the upper bounds:

$$tr(P) \leq t_{U10} = tr(P_U) + \frac{tr(Q + A^T P_U A - P_U)}{1 - \text{±}_n^2(A)} \tag{3.58}$$

$$tr(P) \leq t_{U11} = \frac{tr(Q) - \text{¥}_1(AA^T, P)}{1 - \text{±}_1^2(A)}, \qquad \text{if } A \in \boldsymbol{S}^{\boldsymbol{1}} \tag{3.59}$$

$$tr(P) \leq t_{U12} = \frac{tr(\Phi_1 Q) - \text{¥}_1(\Phi_1 - A\Phi_1 A^T, P)}{\text{ª}_n(\Phi_1 - A\Phi A^T)}, \qquad \text{if } A \in \tilde{\boldsymbol{S}} \tag{3.60}$$

$$tr(P) \leq t_{U13} = \frac{tr(\Phi_2^{-1} Q) - \text{¥}_1(\Phi_2^{-1} - A\Phi_2^{-1} A^T, P)}{\text{ª}_n(\Phi_2^{-1} - A\Phi_2^{-1} A)}, \quad \text{if } A \in \tilde{\boldsymbol{S}} \tag{3.61}$$

where the positive scalar $\text{¥}_1(X, Y)$ is defined in (2.48).

Proof. If P_U satisfies (3.47) it is an upper matrix bound for P in (1.9), or, some positive (semi)-definite matrix Δ_U exists, such that $P = P_U - \Delta_U$ and (1.9) is rewritten as:

$$A^T \Delta_U A - \Delta_U = Q + A^T P_U A - P_U$$

Application of the trace operator to both sides of this DALE results in:

$$-tr(Q + A^T P_U A - P_U) = tr[(I - AA^T)\Delta_U]$$

The right-hand side of the above trace identity can be estimated as follows:

$$
\begin{aligned}
tr[(I-AA^T)\Delta_U] &\le \lambda_1(I-AA^T)tr(\Delta_U)\\
&=[1+\lambda_1(-AA^T)]tr(\Delta_U)\\
&=[1-\lambda_n(AA^T)]tr(\Delta_U)\\
&=[1-\sigma_n^2(A)]tr(\Delta_U)\\
&\Rightarrow \frac{-tr(Q+A^TP_UA-P_U)}{1-\sigma_n^2(A)} \le tr(\Delta_U)
\end{aligned}
$$

This proves the first trace bound (3.58).

Consider the DALE (1.9) and the trace equality:

$$tr(Q)=tr[(I-AA^T)P]$$

Denote $\overline{A}=I-AA^T$. Having in mind (2.47) the right hand-side can be estimated as:

$$
\begin{aligned}
tr[(I-AA^T)P] &\ge \sum_{i=1}^{n}\lambda_{n-i+1}(\overline{A})\lambda_i(P)\\
&=\lambda_n(\overline{A})tr(P)+\sum_{i=2}^{n}[\lambda_{n-i+1}(\overline{A})-\lambda_n(\overline{A})]\lambda_i(P)
\end{aligned}
$$

where

$$\lambda_n(\overline{A})=1-\sigma_1^2(A),\qquad \lambda_{n-i+1}(\overline{A})=1+\lambda_{n-i+1}(-AA^T)=1-\lambda_i(AA^T)=1-\sigma_i^2(A)$$

This results in the inequality:

$$
\begin{aligned}
tr[(I-AA^T)P] &\ge [1-\sigma_1^2(A)]tr(P)+\sum_{i=2}^{n}[\sigma_1^2(A)-\sigma_i^2(A)]\lambda_i(P)\\
&\ge [1-\sigma_1^2(A)]tr(P)+\sum_{i=2}^{n}[\sigma_1^2(A)-\sigma_i^2(A)]l_i\\
&=[1-\sigma_1^2(A)]tr(P)+\yen_1(AA^T,P)
\end{aligned}
$$

This proves the bound (3.59). If $A\in\tilde{\boldsymbol{S}}$, then Φ_1^{-1}, Φ_2 are Lyapunov matrices for *A*, in accordance with Remark 3.2, or equivalently, their inverses are Lyapunov matrices for A^T (see (3.13)). Consider the trace equalities (3.15) and (3.16). Application of the same scheme leads to the proof of the bounds (3.60) and (3.61).

Remark 3.5 The following general conclusions can be drawn from the trace bounds (3.58)-(3.61):

(i) Bound (3.58) is an always tighter upper estimate for the DALE solution trace than $tr(P_U)$ for any satisfying the inequality (3.46) upper matrix bound P_U for *P*. Note, that $A^T P_U A - P_U + Q \le 0$, by definition. If $t_{U10} = tr(P_U)$, then $tr(A^T P_U A - P_U + Q) = 0$, which is possible if and only if $A^T P_U A - P_U + Q = 0$. By uniqueness of the DALE solution it follows that $P = P_U$ and $tr(P) = t_{U10} = tr(P_U)$, in this case.

(ii) By definition, the scalar ¥$_1(X,Y)$ is always nonnegative. Therefore, the following concerning similar bounds relations hold: $(1.42) \ge (3.59), \quad (3.14) \ge \min\{(3.60), (3.61)\}$.

If the lower eigenvalue bounds $l_i \le$ ª$_i(P), i = 2,..,n$ are all positive, the equality sign is possible if and only if

$$±_1(X) = ±_i(X), i = 2,...,n \Leftrightarrow XX^T = ±_1^2(X) I$$

If this is so, the bounds coincide and provide the exact solution trace.

3.4 UNCONDITIONAL MATRIX BOUND

The existence of upper matrix and scalar bounds for the CALE solution which are always valid has been already proved. Following the same approach it will be shown that such unconditional bounds can be derived for the DALE solution, as well.

Consider the Schur decomposition of matrix *A* in (2.77). Having in mind that *A* in (1.9) is a stable in the discrete-time sense matrix, one gets for the diagonal elements of matrix *T* the condition $|$ª$_i(A)|^2 < 1, i = 1,..,n.$ With this preliminary remark the following corresponding to Lemma 2.7 condition can be formulated.

Lemma 3.2 A *n* x *n* matrix *A* is stable in the discrete-time sense if and only if a positive scalar £ ≤ 1 exists, such that $\Xi = U^* E^{-2} U$ is a Lyapunov matrix for *A*, where *E* is a diagonal matrix with entries $e_{ii} =$ £^{i}, $i = 1,..,n.$

Proof. Let A be a stable matrix and consider the matrix

$$A^T \Xi A - \Xi = U^* T^* U (U^* E^{-2} U) U^* T U - U^* E^{-2} U$$
$$= U^* (T^* E^{-2} T - E^{-2}) U = U^* E^{-1} (E T^* E^{-2} T E - I) E^{-1} U$$

Matrix $A^T \Xi A - \Xi$ is negative definite if and only if

$$ET^{*}E^{-2}TE = \Phi(£) = \tilde{\Phi}^{T}(£)\tilde{\Phi}(£) < I \tag{3.62}$$

where the upper triangular matrix $\tilde{\Phi}(£)$ is defined as follows:

$$\tilde{\Phi}(£) = [¤_{ij}],\ ¤_{ij} = \begin{cases} \text{a}_i(A),\ i = j \\ £^{\,j-i} t_{ij},\ i < j \\ 0, \qquad i > j \end{cases}$$

This makes possible to rewrite it as:

$$\tilde{\Phi}(£) = \Lambda + \sum_{k=1}^{n-1} £^{k} T_k$$

where Λ is a diagonal matrix containing the eigenvalues of *A*, and $T_k,\ k = 1,..,n-1$, is an upper triangular matrix containing the off-diagonal entries $t_{ij},\ j - i = k > 0$ of T. Condition (3.62) can be now rewritten as:

$$I > \Phi(£) = (\Lambda^{*} + \sum_{k=1}^{n-1} £^{k} T_k^{*})\ (\Lambda + \sum_{k=1}^{n-1} £^{k} T_k)$$

$$= \Lambda^{*}\Lambda + \sum_{k=1}^{n-1} £^{k} (\Lambda^{*} T_k\ + T_k^{*}\Lambda) + (\sum_{k=1}^{n-1} £^{k} T_k^{*})\ (\sum_{k=1}^{n-1} £^{k} T_k)$$

$$= \Lambda^{*}\Lambda + \sum_{k=1}^{2(n-1)} £^{k} \tilde{T}_k \tag{3.63}$$

Since *A* is a stable matrix, then $I - \Lambda^{*}\Lambda > 0$. Obviously, some sufficiently small $£ \le 1$ always exists, such that (3.63) is satisfied. In particular, if $±_1(A) < 1$, i.e., $A \in \boldsymbol{S}^{\boldsymbol{1}}$, the condition (3.63) holds for $£ = 1$ and $\Xi = U^{*}E^{-2}U = I$ is a Lyapunov matrix for *A*.

Obviously, if (3.63) is satisfied, *A* is a stable matrix. This completes the proof of the Lemma.

Remark 3.6 As in the continuous-time case (see Lemma 2.8), there exist different ways to compute an appropriate $£ \le 1$, i.e. to construct a Lyapunov matrix for A. Note, that for $£ = 1$

$$\sum_{k=1}^{2(n-1)} \tilde{T}_k = \tilde{T},\ T = \Lambda + \tilde{T}\ ,\ \left|\sum_{k=1}^{2(n-1)} T_k\right| = \left|\tilde{T}\right| \tag{3.64}$$

Having in mind (3.63), application of Theorem 2.17 helps to compute $£$ (if $£ \le 1$)from the inequality

$$£ < \frac{1}{° [|\tilde{T}|(I - \Lambda^{*}\Lambda)^{-1}]}, \quad |\tilde{T}| = [|t_{ij}|]$$

which actually corresponds to condition (v) in Lemma 2.8 for the continuous-time case. When appropriately modified, the rest of the sufficient conditions for the satisfaction of (3.63) in Lemma 2.8 can be also applied for the computation of the parameter $£$, which completely defines the Lyapunov matrix Ξ for *A*. As it was already emphasized, having at disposal a Lyapunov matrix for the coefficient matrix guarantees the existence of all types of upper bounds for the CALE and the DALE solutions. The based on Lemmas 2.7 and 3.2 bounds are always computable, in sense that no additional restrictive suppositions are made with respect to the coefficient matrix A, and they depend entirely on the parameters of the respective equation, and no additional computational procedures, like LMI solution, is needed, in this case.

3.5 FURTHER EXTENSION OF VALIDITY SETS

The application of the singular decomposition approach helps to extend the set of the stable matrices for which computable upper bounds for the solution of DALE exist. This resulted in the definition of the less conservative matrix $\tilde{\boldsymbol{S}}$. It has been shown how the fact that matrix *A* belongs to this set can be used to derive upper bounds under less restrictive conditions for validity. Now, in an attempt to decrease further this conservatism, we shall try to extend the validity sets $\boldsymbol{S}^1$ and $\tilde{\boldsymbol{S}}$. Consider a simple motivating example.

Example 3.2 Let the following matrix be given:

$$A = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}, \quad ±_1^2(A) = a^2 + b^2, \quad ª_1(A) = b, \quad ª_2(A) = ±_2(A) = 0$$

Obviously, A is a stable matrix if $|b| < 1$. Suppose that $a^2 + b^2 = 1$, which guarantees stability of *A*, but $±_1(A) = 1$**,** in this case**,** and none of the upper bounds for the DALE solution requiring $A \in \boldsymbol{S}^1$ can be used, in this simple case. Consider now matrix

$$A^2 = \begin{bmatrix} 0 & ab \\ 0 & b^2 \end{bmatrix}, \quad ±_1^2(A^2) = b^2(a^2 + b^2) = b^2$$

Therefore, if A is a stable matrix, then $\pm_1 (A^2) = |b| < 1$, or $A^2 \in \boldsymbol{S^1}$.

It will be shown how this fact can be used to extend the conservative matrix set $\boldsymbol{S}^1$. Also, an attempt to extend the set $\tilde{\boldsymbol{S}}$ will be made. Before that, all important results concerning the conditions for validity of upper bounds obtained in this Chapter will be briefly summarized.

The sets $\boldsymbol{S}^1$ and $\tilde{\boldsymbol{S}}$ are characterized by the conditions $\pm_1 (A) < 1$ and $ª_1(R_1 R_2) < 1$ (see Lemma 3.1), respectively, where matrices R_1 and R_2 are defined in (3.10). Since $\pm_1(A) = ª_1(R_1) = ª_1(R_2)$, the first condition can be represented in terms of matrices R_1 and R_2 as:

$$1 > \pm_1(A) \Leftrightarrow 1 > \pm_1^2(A) = ª_1(R_1)ª_1(R_2)$$

It has been proved in Theorem 3.1 that $\boldsymbol{S^1} \subseteq \tilde{\boldsymbol{S}}$, and the above inequality illustrates also this fact, since $ª_1(R_1)ª_1(R_2) \geq ª_1(R_1 R_2)$.

Consider the matrix A^2. Having in mind the singular value decomposition of A in (3.1), (3.2) and the notation (3.3), one gets:

$$\begin{aligned}
\pm_i(A^2) &= \pm_i(U\Sigma V^T U\Sigma V^T) \\
&= \pm_i(\Sigma V^T U\Sigma) \\
&= \pm_i(V\Sigma V^T U\Sigma U^T) \\
&= \pm_i(R_1 R_2), \; i = 1,..,n
\end{aligned}$$

$$\begin{aligned}
rank(A^2) &= rank(\Sigma V^T U \Sigma) \\
&= rank\{\begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}\} \\
&= rank\{\begin{bmatrix} \Xi_r F_{11} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{bmatrix}\} \\
&= rank(\Xi_r F_{11} \Xi_r) \\
&= rank(\Xi_r F_{11}) \\
&= rank(F_{11}) \\
&= f \leq r
\end{aligned}$$

It follows that the singular value decomposition of A^2 is given by:

$$A^2 = \tilde{U}\tilde{\Sigma}\tilde{V}^T, \quad \tilde{U}\tilde{U}^T = \tilde{V}\tilde{V}^T = I\,, \quad \tilde{\Sigma} = \begin{bmatrix} \tilde{\Xi}_f & 0_{f,n-f} \\ 0_{n-f,f} & 0_{n-f} \end{bmatrix} \tag{3.65}$$

where $f = rank(A^2) \le r = rank(A) \le n$ and the diagonal matrix $\tilde{\Xi}_f$ contains the positive singular values $\pm_i(A^2) = \pm_i(R_1R_2),\, i = 1,..,f$, of A^2 .

Denote

$$\tilde{R}_1 = [A^2(A^2)^T]^{1/2}, \quad \tilde{R}_2 = [(A^2)^T A^2]^{1/2} \tag{3.66}$$

$$\tilde{\Phi}_1 = U\bar{\Sigma}_1 U^T, \quad \bar{\Sigma}_1 = \begin{bmatrix} \tilde{\Xi}_f & 0_{f,n-f} \\ 0_{n-f,f} & \circ_1^2 I_{n-f} \end{bmatrix}, \quad \circ_1 > 0 \tag{3.67}$$

$$\tilde{\Phi}_2 = V\bar{\Sigma}_2 V^T, \quad \bar{\Sigma}_2 = \begin{bmatrix} \Xi_r & 0_{r,n-r} \\ 0_{n-r,r} & \circ_2^2 I_{n-r} \end{bmatrix}, \quad \circ_2 > 0, \tag{3.68}$$

Lemma 3.3 X is a Lyapunov matrix for A^k, $k = 2, 3..$, if and only if

$$X_E = \sum_{i=0}^{k-1} (A^i)^T X A^i$$

is a Lyapunov matrix for A.

Proof. It follows from the matrix equality

$$(A^k)^T X A^k - X = A^T[\sum_{i=0}^{k-1}(A^i)^T X A^i]A - \sum_{i=0}^{k-1}(A^i)^T X A^i = A^T X_E A - X_E$$

Lemma 3.4 Define the extended matrix sets:

$$\boldsymbol{S}_E^1 \equiv \{A,\, A \in \boldsymbol{R}_n : \pm_1(A^2) < 1\}$$

$$\tilde{\boldsymbol{S}}_E \equiv \{A,\, A \in \boldsymbol{R}_n : ª_1(\tilde{R}_1\tilde{R}_2) < 1\}$$

(i)The following statements are equivalent:

(a) $A \in \boldsymbol{S}_E^1$

(b) $\pm_1(R_1R_2) < 1$, where matrices R_1, R_2 are defined in (3.10)

(c) $ª_1(\tilde{R}_1)ª_1(\tilde{R}_2) < 1$, where matrices $\tilde{R}_1$ and $\tilde{R}_2$ are defined in (3.66)

(d) $P_E = A^T A + I$ is a Lyapunov matrix for A

(ii) The following statements are equivalent:

(a) $A \in \tilde{\boldsymbol{S}}_{\boldsymbol{E}}$

(b) there exist some positive scalars $°_1$, $°_2$, such that

$$P_{E1} = A^T \tilde{\Phi}_1^{-1} A + \tilde{\Phi}_1^{-1} \tag{3.69}$$

$$P_{E2} = A^T \tilde{\Phi}_2 A + \tilde{\Phi}_2 \tag{3.70}$$

are Lyapunov matrices for *A*, where $\tilde{\Phi}_1, \tilde{\Phi}_2$ are given in (3.67), (3.68).

(iii) $\boldsymbol{S^1} \subseteq \boldsymbol{S_E^1} \subseteq \tilde{\boldsymbol{S}}_{\boldsymbol{E}}$

Proof Statement (i):

(a)$\Rightarrow$(b) Suppose that $A \in \boldsymbol{S_E^1}$. From (3.65) it follows that $\pm_1(A^2) = \pm_1(R_1R_2) < 1$.

(b) $\Rightarrow$(c) Let $\pm_1(A^2) < 1$. Having in mind (3.66), one obtains $ª_1(\tilde{R}_1) = ª_1(\tilde{R}_2) = \pm_1(A^2)$, which implies statement (c).

(c) $\Rightarrow$(d) If (c) holds, then the proof follows from Lemma 3.3 applied for $k = 2$

(d) $\Rightarrow$(a) Due to Lemma 3.2, this implication is obvious.

This completes the proof of statement (i)

(ii) Statement (ii):

(a) $\Rightarrow$(b) If $A \in \tilde{\boldsymbol{S}}_{\boldsymbol{E}}$, then application of statement (ii) in Theorem 3.1 for A^2 with singular value decomposition (3.66) and $\tilde{\Phi}_1, \tilde{\Phi}_2$ given in (3.67), (3.68) guarantees that $\tilde{\Phi}_1^{-1}$ and $\tilde{\Phi}_2$ are Lyapunov matrices for A^2, in this case. From Lemma 3.3 applied for $X = \tilde{\Phi}_1^{-1}$, $X = \tilde{\Phi}_2$ and $k = 2$, one gets (b).

(b) $\Rightarrow$(a) According to Theorem 3.1, statement (ii), but now applied for A^2 it follows, that $A \in \tilde{\boldsymbol{S}}_{\boldsymbol{E}}$.

This completes the proof of statement (ii).

(iii) Let $A \in \boldsymbol{S^1}$, then

$$1 > ª_1(R_1)ª_1(R_2) = \pm_1(R_1)\pm_1(R_2) \geq \pm_1(R_1R_2)$$

and in accordance with statement (i), (b) it follows, that $A \in \boldsymbol{S_E^1}$. Let $A \in \boldsymbol{S_E^1}$, which is equivalent to the inequality $ª_1(\tilde{R}_1)ª_1(\tilde{R}_2) < 1$, in accordance with statement (i), (c). Since $ª_1(\tilde{R}_1)ª_1(\tilde{R}_2) \geq ª_1(\tilde{R}_1\tilde{R}_2)$, it follows that $A \in \tilde{\boldsymbol{S}}_{\boldsymbol{E}}$.

This completes the proof of the Lemma.
Obviously, the process of set extension for stable matrices for which computable upper bounds for the DALE solution can be continued by considering the cases $A^k, k > 2$. E.g. Lemma 3.3 provides exact condition under which a Lyapunov matrix for *A* exists, if a Lyapunov matrix for A^k has been defined. The observed decrease in conservatism is due to the fact that as the integer *k* increases, one gets less and less restrictive requirements for validity of all upper bounds considered here.

3.6 BOUNDS FOR THE DARE

Various scalar, matrix, lower and upper bounds for the positive definite solution of the DARE (1.14) have been suggested. With only two exceptions, they require positive definiteness of matrix *Q* for lower bounds (e.g. (1.43), (1.49)) and positive definiteness of matrix BB^T for upper bounds (e.g. (1.44), (1.45), (1.48), (1.50)). The first exception is due to the lower matrix bound (1.59), which is always applicable. The other one is the upper bound in (1.59), which is valid for BB^T singular, but requires stability of the coefficient matrix *A*, which is not realistic, as well.
The fact that (*A*, *B*) is a stabilizable pair by assumption (see Theorem 1.5) is important, but usually it is not taken into account. An attempt to overcome to a certain extent the main stated difficulties will be made in this part.
Having in mind Remark 1.1, consider the DARE (1.14) rewritten as follows:

$$(A^T - K^T B^T)P(A - BK) - P - A^T PBS^{-1}B^T PA + K^T B^T PA + A^T PBK - K^T B^T PBK = -Q$$

where $K \in \boldsymbol{R}_{m,n}$, and $S = I + B^T PB$. Using the notation $A_C = A - BK$, and by adding and subtracting $K^T B^T SBK$ in the left-hand side of the equation, the DARE takes the form

$$A_C^T PA_C - P - (A^T PB - K^T B^T S)S^{-1}(B^T PA - SBK) + K^T B^T SBK - K^T B^T PBK = -Q$$

It can be compactly represented as a DALE-type:

$$A_C^T PA_C - P = \tilde{S} - K^T K - Q \qquad (3.71)$$

where $\tilde{S} = (A^T PB - K^T B^T S)S^{-1}(B^T PA - SBK)$. The pair (A, B) is stabilizable, and therefore some matrix K always exists, such that $A_C = A - BK$ is a stable in the discrete-time sense matrix, i.e. from now on we shall assume that A_C in (3.71) is stable.

Lemma 3.5 The solution P of the DARE (1.14) is bounded from above by the solution U of the DALE

$$A_C^T U A_C - U + \overline{Q} = 0,\ \ \overline{Q} = K^T K + Q \qquad (3.72)$$

and, consequently, by any upper bound for it.

Proof. Matrix $\tilde{S}$ is positive (semi)-definite, and (3.71) implies the inequality

$$A_C^T P A_C - P + \overline{Q} \geq 0,\ \ \overline{Q} = K^T K + Q$$

Therefore,

$$A_C^T P A_C - P + \overline{Q} \geq A_C^T U A_C - U + \overline{Q}$$

$$\Rightarrow A_C^T (U - P) A_C - (U - P) \leq 0$$

Since A_C is a stable matrix, this is possible only if $U \geq P$. Obviously, if $P_U \geq U$, then P_U is also an upper bound for the DARE solution P.

Theorem 3.11 The solution of the DARE (1.14) has the following upper bounds:

$$P \leq P_{U1} = «_{U1}\Phi_1^{-1},\quad «_{U1} = ª_1[\overline{Q}(\Phi_1^{-1} - A_C^T \Phi_1^{-1} A_C)^{-1}],\qquad \text{if } A_C \in \tilde{\boldsymbol{S}} \quad (3.73)$$

$$P \leq P_{U2} = «_{U2}\Phi_2,\quad «_{U2} = ª_1[\overline{Q}(\Phi_2 - A_C^T \Phi_2 A_C)^{-1}],\qquad \text{if } A_C \in \tilde{\boldsymbol{S}} \quad (3.74)$$

where $\overline{Q} = K^T K + Q$, Φ_1^{-1}(3.4) and Φ_2(3.5) are Lyapunov matrices for A_C in accordance with statement (iii) in Lemma 3.1,

$$P \leq P_{U,p} = \overline{Q} + ª_1(\overline{Q})[\sum_{i=1}^{p-1}(A_C^i)^T A_C^i + \frac{1}{1 - ±_1^2(A_C)}(A_C^p)^T A_C^p] - S(a,q),\ \ \text{if } A_c \in \boldsymbol{S}^1 \quad (3.75)$$

for arbitrary $p = 1,2,...,\ a,q = 0,1,2,...$, where

$$S(a,q) = A_C(a) + \overline{Q}(q)$$

$$A_C(a) = ª_1(\overline{Q})\sum_{i=1}^{a}\sum_{s=0}^{i-1} ±_1^{2(i-1-s)}(A_C^{s+1})^T \Delta_{A_C} A_C^{s+1},\ \ \overline{Q}(q) = \sum_{i=1}^{q}(A_C^i)^T \Delta_{\overline{Q}} A_C^i$$

$$\Delta_A = ±_1^2(A_C)I - A_C^T A_C,\quad \Delta_{\overline{Q}} = ª_1(\overline{Q})I - \overline{Q}$$

$$P \le \overline{P}_{U,p} = \overline{Q} + \text{ª}_1(\tilde{Q})[\sum_{i=1}^{p-1}(A_C^i)^T T^{-2} A_C^i + \frac{1}{1-\pm_1^2(\tilde{A}_C)}(A_C^p)^T T^{-2} A_C^p] - \overline{S}(a,q)\text{, if } \tilde{A}_c \in \boldsymbol{S^1} \quad (3.76)$$

for arbitrary $p = 1,2,...,\ a,q = 0,1,2,...$, where $\tilde{A}_C = T^{-1}A_C T,\ \tilde{Q} = T\overline{Q}T$,

$$\overline{S}(a,q) = \overline{A}_C(a) + \overline{Q}(q)$$

$$\overline{A}(a) = \text{ª}_1(\tilde{Q})\sum_{i=1}^{a}\sum_{s=0}^{i-1}\pm_1^{2(i-1-s)}(A_C^{s+1})^T\overline{\Delta}_A A_C^{s+1}, \quad \overline{\Delta}_A = \pm_1^2(\tilde{A}_C)T^{-2} - A_C^T T^{-2} A_C$$

$$\overline{Q}(q) = [\sum_{i=1}^{q}(A_C^i)^T\overline{\Delta}_Q A_C^i], \quad \overline{\Delta}_{\overline{Q}} = \text{ª}_1(\tilde{Q})T^{-2} - \overline{Q}$$

$$P \le P_{U1,p} = \overline{P}_{U,p}, \quad T^2 = \Phi_1, \qquad \text{if } A_C \in \tilde{\boldsymbol{S}} \quad (3.77)$$

$$P \le P_{U2,p} = \overline{P}_{U,p}, \quad T^2 = \Phi_2^{-1}, \qquad \text{if } A_C \in \tilde{\boldsymbol{S}} \quad (3.78)$$

for arbitrary $p = 1,2,...,\ a,q = 0,1,2,...$, where the upper bound $\overline{P}_{U,p}$ is given in (3.76), Φ_1^{-1} (3.4) and Φ_2 (3.5) are Lyapunov matrices for A_C in accordance with statement (iii) in Lemma 3.1. In addition, for all p = 1, 2,.., one has

$$P_{U,p} \le P_{U,p-1}, \quad P_{U1,p} \le P_{U1,p-1}, \quad P_{U2,p} \le P_{U2,p-1} \qquad (3.79)$$

Proof. Having in mind Lemma 3.5, the proof of the upper matrix bounds (3.73)-(3.78) is entirely based on Theorem 3.8 (for bounds (3.73), (3.74)), Theorem 3.6 (for bound (3.75)), Theorem 3.7 (for bound (3.76), and Corollary (3.1) (for bounds (3.77) and (3.78)), when the respective statements are applied for A_C and $\overline{Q}$ in (3.72) instead of A and Q in (1.9). The same refers to the matrix inequalities (3.79).

Remark 3.7 Due to Lemma 3.5, it is possible to estimate the DARE solution from above by the upper bounds for a respective DALE solution. In other words, it is very important to have less restrictive with respect to validity upper bounds for the DALE solution matrix. The bounds (3.73), (3.74), (3.77) and (3.78) are preferable to the estimates (3.75) and (3.76). Moreover, the bounds (3.73), (3.74), (3.77), (3.78) are valid whenever the bound (3.75) is computable. When the upper estimate (3.76) is obtained as a result of the computation of a satisfying the validity condition $\tilde{A}_c \in \boldsymbol{S^1}$ unknown matrix T, it is an external bound. The upper matrix bounds given above can be used to derive upper bounds for the maximal solution, i.e.

$$\lambda_1(P) \le e_{U1} = \min\{\lambda_1(P_{U1}), \lambda(P_{U2})\} \;, \qquad \text{if } A_C \in \tilde{\boldsymbol{S}} \quad (3.80)$$

$$\lambda_1(P) \le e_{U2,p} = \lambda_1(P_{U,p}), \quad \forall p \;, \qquad \text{if } A_c \in \boldsymbol{S}^{\boldsymbol{1}} \quad (3.81)$$

$$\lambda_1(P) \le e_{U3,p} = \lambda_1(\overline{P}_{U,p}), \quad \forall p \;, \qquad \text{if } \tilde{A}_c \in \boldsymbol{S}^{\boldsymbol{1}} \quad (3.82)$$

$$\lambda_1(P) \le e_{U4,p} = \min\{\lambda_1(P_{U1,p}), \lambda_1(P_{U2,p})\}, \quad \forall p \;, \qquad \text{if } A_C \in \tilde{\boldsymbol{S}} \quad (3.83)$$

and for the trace of the DARE solution:

$$tr(P) \le t_{U1} = \min\{tr(P_{U1}), tr(P_{U2})\} \;, \qquad \text{if } A_C \in \tilde{\boldsymbol{S}} \quad (3.84)$$

$$tr(P) \le t_{U2,p} = tr(P_{U,p}), \quad \forall p \;, \qquad \text{if } A_c \in \boldsymbol{S}^{\boldsymbol{1}} \quad (3.85)$$

$$tr(P) \le t_{U3,p} = tr(\overline{P}_{U,p}), \quad \forall p \;, \qquad \text{if } \tilde{A}_c \in \boldsymbol{S}^{\boldsymbol{1}} \quad (3.86)$$

$$tr(P) \le t_{U4,p} = \min\{tr(P_{U1,p}), tr(P_{U2,p})\}, \quad \forall p \;, \qquad \text{if } A_C \in \tilde{\boldsymbol{S}} \quad (3.87)$$

where matrices $P_{U1}, P_{U2},\ P_{U,p}, \overline{P}_{U,p}, P_{U1,p}, P_{U2,p}$ are defined in Theorem 3.11.

The upper matrix bounds due to Theorem 3.11 are obtained in accordance with Lemma 3.5, which states that the DARE solution can be bounded from above by the respective upper estimates for the solution of the DALE (3.72). This means that under the respective conditions for validity and having in mind that the matrix pair (*A*, *Q*) must be suitably replaced with the pair (A_C , $\overline{Q}$), all upper bounds for the DALE (3.73) are actually bounds for the DARE solution (see, e.g. Theorems 3.2 and 3.3).

Besides the fact that due to the inequalities in (3.79) the respective eigenvalue (3.80)-(3.83) and trace (3.84)-(3.87) bounds are getting tighter and tighter as the parameter *p* increases, some upper DARE estimates can be improved in sense of tightness, as well. Consider the upper matrix bounds (3.73) and (3.74). They are based on inequality (3.47), which in these particular cases takes the form

$$A_C^T P_{U1} A_C - P_{U1} + \overline{Q} \le 0 \;,\; A_C^T P_{U2} A_C - P_{U2} + \overline{Q} \le 0 \qquad (3.88)$$

This means that the bounds P_{U1}, P_{U2} for the DARE solution can be improved in sense of tightness in accordance with Corollary 3.5, i.e. for any $i = 1, 2, \ldots$, one gets the matrix bounds:

$$P \le U_{1i} = (A_C^i)^T P_{U1} A_C^i + \sum_{j=0}^{i-1} (A_C^j)^T \overline{Q} A_C^j \;, \qquad U_{1_i} \le U_{1_{i-1}} \qquad (3.89)$$

$$P \le U_{2i} = (A_C^i)^T P_{U2} A_C^i + \sum_{j=0}^{i-1} (A_C^j)^T \overline{Q} A_C^j \,, \qquad U_{2_i} \le U_{2_{i-1}} \tag{3.90}$$

This helps to get respective scalar bounds:

$$ª_1(P) \le e_{U5,i} = \min\{ª_1(U_{1_i}),\ ª_1(U_{2_i})\}\,, \qquad e_{U5,i} \le e_{U5,i-1} \tag{3.91}$$

$$tr(P) \le t_{U5,i} = \min\{tr(U_{1_i}),\ tr(U_{2_i})\} \qquad t_{U9,i} \le t_{U9,i-1} \tag{3.92}$$

Having in mind that $\tilde{S}$ in (3.71) is a positive (semi)-definite matrix, and

$$tr(\overline{Q} - S) \le tr(\overline{Q}),\ \ \overline{Q} = Q + K^T K$$

application of Theorem 3.10 for (*A*, *Q*) replaced with (A_C , $\overline{Q}$) in (3.72) results in the following tighter upper trace bounds for the DARE solution *P*:

$$tr(P) \le t_{U6} = tr(P_U) + \frac{tr(\overline{Q} + A_C^T P_U A_C - P_U)}{1 - ±_n^2(A_C)} \tag{3.93}$$

$$tr(P) \le t_{U7} = \frac{tr(\overline{Q}) - ¥_1(A_C A_C^T, P)}{1 - ±_1^2(A_C)}\,, \qquad \text{if } A \in \boldsymbol{S}^1 \tag{3.94}$$

$$tr(P) \le t_{U7} = \frac{tr(\Phi_1 \overline{Q}) - ¥_1(\Phi_1 - A_C \Phi_1 A_C^T, P)}{ª_n(\Phi_1 - A_C \Phi A_C^T)}\,, \qquad \text{if } A \in \tilde{\boldsymbol{S}} \tag{3.95}$$

$$tr(P) \le t_{U8} = \frac{tr(\Phi_2^{-1} Q) - ¥_1(\Phi_2^{-1} - A\Phi_2^{-1} A^T, P)}{ª_n(\Phi_2^{-1} - A\Phi_2^{-1} A)}\,, \qquad \text{if } A \in \tilde{\boldsymbol{S}} \tag{3.96}$$

where $P_U = P_{U1}, P_{U2}$ are defined in (3.74) and (3.75), respectively, $l_i \le ª_i(P), i = 2,..,n$, the positive scalar $¥_1(X,Y)$ is defined in (2.48) and Φ_1^{-1} (3.4) and Φ_2 (3.5) are Lyapunov matrices for A_C in accordance with statement (iii) in Lemma 3.1. The nonnegative scalars $l_i, i = 2,..,n$ can be chosen as the eigenvalues of the given in (1.59) lower matrix bound for *P*, which holds for any positive semi-definite matrix *Q*. Note, that $P \ge Q$, which means that $l_i = ª_i(Q), i = 2,..,n$, is an admissible choice, as well.

CHAPTER FOUR

NUMERICAL EXAMPLES

The applicability of the suggested bounds for the four algebraic equations will be illustrated by numerical examples and compared with available similar bounds with respect to validity and tightness. Before that, define the percentage errors in lower and upper scalar bounds as follows:

$$\varepsilon_L(b) = (1 - \frac{\text{lower bound}}{\text{exact value}})\,\text{x}\,100 \tag{4.1}$$

$$\varepsilon_U(b) = (\frac{\text{upper bound}}{\text{exact value}} - 1)\,\text{x}\,100 \tag{4.2}$$

Also, the nonnegative scalars

$$d = \min d_i, \quad D = \max d_i, \quad d_i = \lambda_i(P_U - P_L),\, i = 1,..,n \tag{4.3}$$

$$t_e(P_U, P_L) = tr(P_U - P_L) \tag{4.4}$$

$$s_i = \frac{1}{2}\lambda_i(P_U + P_L),\, i = 1,..,n, \quad t_a(P_U, P_L) = \frac{1}{2}tr(P_U + P_L) \tag{4.5}$$

will be used to evaluate the quality of the respective bounds, where P_L, P_U are a lower and an upper matrix bounds for *P*, respectively. In fact, *d* and *D* represent the minimal and the maximal eigenvalues of the "error" matrix, and s_i denotes the *i*-th eigenvalue of the "average estimation" matrix. The "average" eigenvalue of an *n* x *n* symmetric matrix *M* is defined as

$$\lambda_a(X) = \frac{tr(X)}{n} \tag{4.6}$$

4.1 THE CALE BOUNDS

Example 4.1.1 Fifth order industrial reactor [6]

Consider a stable coefficient matrix:

$$A=\begin{bmatrix} -16.11 & -0.39 & 27.2 & 0 & 0 \\ 0.01 & -16.99 & 0 & 0 & 12.47 \\ 15.11 & 0 & -53.6 & -16.57 & 71.78 \\ -53.36 & 0 & 0 & -107.2 & 232.11 \\ 2.27 & 60.1 & 0 & 2.273 & -102.99 \end{bmatrix}$$

For $Q = I_5$, the trace and the eigenvalues of the positive definite solution for the CALE (1.2) are: $tr(P) = 0.4859$ and

$$\lambda_1(P) = 0.3473,\ \lambda_2(P) = 0.1115,\ \lambda_3(P) = 0.0172,\ \lambda_4(P) = 0.00724,\ \lambda_5(P) = 0.0026$$

Although A is a stable matrix, its symmetric part is not negative definite ($\lambda_1(A_S) = 17.1$), and therefore, none of the based on the condition $A \in \boldsymbol{H}^-$ upper bounds can be used in this case.

Lower bounds for the minimal eigenvalue and the trace of P:

$$\lambda_5(P) = 0.0026 \geq \begin{cases} 0.0017, & \text{bound (1.16)} \\ 0.0011, & \text{bound (1.17)} \\ 0.0022, & \text{bound (1.25)} \\ 0.0022, & \text{bound (1.74)} \end{cases}, \quad \begin{matrix} \varepsilon_L = 34.60\% \\ \varepsilon_L = 57.7\% \\ \varepsilon_L = 15.38\% \\ \varepsilon_L = 15.38\% \end{matrix} \tag{4.7}$$

$$tr(P) = 0.4859 \geq \begin{cases} 0.0108, & \text{bound (1.20)} \\ 0.0084, & \text{bound (1.21)} \\ 0.0421, & \text{bound (1.22)} \end{cases}, \quad \begin{matrix} \varepsilon_L = 97.8\% \\ \varepsilon_L = 98.3\% \\ \varepsilon_L = 91.3\% \end{matrix} \tag{4.8}$$

i.e.

$$\lambda_5(P) \geq 0.0022, \quad tr(P) \geq 0.0421 \tag{4.9}$$

Using the obtained here bounds the following estimates have been computed:

(i) Lower bounds:

$$\lambda_5(P) = 0.0026 \geq \begin{cases} 0.0022, & \text{bound (2.19)} \\ 0.0017, & \text{bound (2.21)} \\ 0.0022, & \text{bound (2.71)} \end{cases}, \quad \begin{matrix} \varepsilon_L = 15.38\% \\ \varepsilon_L = 34.60\% \\ \varepsilon_L = 15.38\% \end{matrix} \tag{4.10}$$

where $\mu = 34.6$ in (2.71),

$$tr(P) = 0.4859 \geq \begin{cases} 0.3349, & \text{bound (2.16)} \qquad \delta_L = 31.07\% \\ 0.4071, & \text{bound (2.35)}, \qquad \delta_L = 16.22\% \\ 0.4334, & \text{bound (2.35)} \qquad \delta_L = 10.80\% \end{cases} \tag{4.11}$$

where the two bounds in (2.35) represent the traces of the lower matrix bounds L_1, L_2, with $P_L = \alpha_L R_1^{-1}$, $\alpha_L = 0.5$, $R_1 = (AA^T)^{1/2}$ (see Theorems 2.12 and 2.13), i.e.

$$\lambda_5(P) \geq 0.0022, \quad tr(P) \geq 0.4334 \tag{4.12}$$

Although $A \notin \boldsymbol{H}^-$, the orthogonal matrix F in (2.1) is stable ($\max \operatorname{Re} \lambda(F) = -0.2955$), i.e. $A \in \tilde{\boldsymbol{H}}$ and all upper bounds based on this condition are computable, in this case.

(ii) Upper bounds:

$$\lambda_1(P) = 0.3473 \leq \begin{cases} 0.4246, & \text{bound (2.17)} \qquad \delta_U = 22.26\% \\ 0.4447, & \text{bound (2.42)} \qquad \delta_U = 28.04\% \\ 0.3967, & \text{bound (2.42)} \qquad \delta_U = 14.22\% \\ 0.3876, & \text{bound (2.42)} \qquad \delta_U = 11.60\% \end{cases}, \tag{4.13}$$

where the three bounds in (2.42) represent the maximal eigenvalues of the upper matrix bounds U_1, U_2, U_3, with $P_U = \alpha_U R_1^{-1}$, $\alpha_U = 1.6918$, $R_1 = (AA^T)^{1/2}$ (see Theorems 2.11 and 2.13).

$$tr(P) = 0.4859 \leq \begin{cases} 1.1331, & \text{bound (2.15)} \qquad \delta_U = 133\% \\ 0.6948, & \text{bound (2.43)} \qquad \delta_U = 43\% \\ 0.6095, & \text{bound (2.43)} \qquad \delta_U = 25.44\% \\ 0.5802, & \text{bound (2.43)} \qquad \delta_U = 19.41\% \end{cases}, \tag{4.14}$$

where the three bounds in (2.43) represent the traces of the upper matrix bounds U_1, U_2, U_3, with $P_U = \alpha_U R_1^{-1}$, $\alpha_U = 1.6918$, $R_1 = (AA^T)^{1/2}$ (see Theorems 2.11 and 2.13), i.e.

$$\lambda_1(P) \leq 0.3876, \quad tr(P) \leq 0.5802 \tag{4.15}$$

From (4.12) and (4.15) one gets the following lower and upper bounds for the extremal eigenvalues and the trace of the solution matrix in (1.2):

$$\lambda_5(P) \geq 0.0022, \quad \lambda_1(P) \leq 0.3876, \quad 0.4334 \leq tr(P) \leq 0.5802 \tag{4.16}$$

Finally, consider the indicators in (4.3)-(4.6), computed for the lower matrix bound L_2 and the upper matrix bound U_3 as follows:

$$d = 0.004, \quad D = 0.12, \quad t_e(L_2, U_3) = 0.176$$

$$s_1 = 0.357,\ s_2 = 0.13,\ s_3 = 0.0165,\ s_4 = 0.0142,\ s_5 = 0.0038;\ t_a(L_2, U_3) = 0.5214$$

$$\lambda_a(L_2) = 0.0086,\quad \lambda_a(U_3) = 0.116$$

Conclusions. Having in mind the obtained results, the following conclusions can be drawn.
(i) Due to the application of the singular value decomposition approach all kinds of upper bounds for the CALE solution are computable, while the based on the conservative condition $A \in \boldsymbol{H}^-$ available estimates are inapplicable, in this case.
(ii) From (4.9) and (4.16) it becomes clear, that the presented here bounds are tighter than the existing similar estimates.
(iii) The eigenvalue "error" varies between d = 0.004 and D = 0.12 and the trace "error" is $t_e(L_2, U_3) = 0.176$. The indices s_i, $i = 1,..,5$, and $t_a(L_2, U_3)$ are very close to the solution eigenvalues and trace, which means that the respective bounds are rather satisfactory. E.g. $s_1 - \lambda_1(P) = 0.0097$, $s_2 - \lambda_2(P) = 0.018$, etc., $t_a(L_2, U_3) - tr(P) = 0.035$.
The "average" solution eigenvalue

$$\lambda_a(P) = \frac{tr(P)}{5} = 0.0972$$

belongs to the tight interval

$$0.0867 = \frac{tr(L_2)}{5} \le \lambda_a(P) \le \frac{tr(U_3)}{5} = 0.116$$

with lower and upper error 10.8% and 19.3%, respectively.

Example 4.1.2 Six-plate gas absorber [28]
The stable coefficient matrix in (1.2) is given by:

$$A = \begin{bmatrix} -1.173 & 0.6341 & 0 & 0 & 0 & 0 \\ 0.539 & -1.173 & 0.6341 & 0 & 0 & 0 \\ 0 & 0.539 & -1.173 & 0.6341 & 0 & 0 \\ 0 & 0 & 0.539 & -1.173 & 0.6341 & 0 \\ 0 & 0 & 0 & 0.539 & -1.173 & 0.6341 \\ 0 & 0 & 0 & 0 & 0.539 & -1.173 \end{bmatrix}$$

The symmetric part of A is negative definite ($\lambda_1(A_S) = -0.11161$) and the same refers to the symmetric part of the orthogonal matrix F ($\max \operatorname{Re} \lambda(F) = -0.99 = \lambda_1(F_S)$, i.e. $A \in \tilde{\boldsymbol{H}}$, by necessity. This means that all upper bounds based on the condition $A \in \boldsymbol{H}^-$ are also applicable, in this case. For Q selected as an 6 x 6 identity matrix, the solution's eigenvalues and trace have been computed as:

$$\lambda_1(P) = 4.2635,\ \lambda_2(P) = 1.1333,\ \lambda_3(P) = 0.5485,$$

$$\lambda_4(P) = 0.3487,\ \lambda_5(P) = 0.2626\ \ \lambda_6(P) = 0.2242,\ tr(P) = 6.7809.$$

(i) Lower bounds:

$$\lambda_6(P) = 0.2242 = \text{bound (1.16)} = \text{bound (1.17)} = \text{bound (1.18)} = \text{bound (1.74)} \qquad (4.17)$$

$$tr(P) = 6.7809 \geq \begin{cases} 1.3453, & \text{bound (1.20)} \quad ¢_L = 80.16\% \\ 0.8525, & \text{bound (1.21)}, \quad ¢_L = 87.43\% \\ 1.3453, & \text{bound (1.22)} \quad ¢_L = 87.43\% \end{cases}$$

The coincidence of three lower bounds for the minimal eigenvalue of P with the exact value is a remarkable fact. The lower trace bounds are not tight at all.

(ii) Upper bounds:

$$\lambda_1(P) = 4.2635 \leq 4.3076 \text{ bound (1.19)}, \quad ¢_U = 1.0344\%$$

$$tr(P) = 6.7809 \leq 6.8236 \text{ bound (1.23)}, \quad ¢_U = 0.63\%$$

Both upper bounds are very sharp.

The following estimates for the CALE solution have been obtained using the suggested here bounds.

(i) Lower bounds:

$$\lambda_6(P) = 0.2242 = \text{bound (2.18)}$$

$$tr(P) = 6.7809 \geq 6.7357 \text{ bound (2.16)}, \quad ¢_L = 0.67\%$$

(ii) Upper bounds:

$$\lambda_1(P) = 4.2635 \leq 4.2763 \text{ bound (2.17)}, \quad ¢_U = 0.3\%$$

$$tr(P) = 6.7809 \leq 6.8046 \text{ bound (2.6)}, \quad ¢_U = 0.35\%$$

The computed for the lower matrix bound $P_{L1} = \alpha_L R_1^{-1}$, $\alpha_L = 0.500072$, in (2.32) and the upper matrix bound $P_{U3} = \alpha_{U3} R_1^{-1}$, $\alpha_{U3} = 0.500505588$ in (2.27), estimation indicators in (4.3)-(4.6), are:

$$d = 0.0023, \quad D = 0.0422, \quad t_e(P_{L1}, P_{U3}) = 0.0671$$

$$s_1 = 4.2552,\ s_2 = 1.2786,\ s_3 = 0.5490,\ s_4 = 0.3490,\ s_5 = 0.26373,\ s_6 = 0.22534;$$

$$t_a(P_{L1}, P_{U3}) = 6.771039$$

$$\lambda_a(P_{L1}) = 1.123, \quad \lambda_a(U_3) = 1.341$$

Conclusions. The obtained via the singular value decomposition approach bounds are again tighter. All lower and upper estimates produce almost the exact values. This fact is confirmed by the values obtained for the error indicators and the “average” solution estimates. Also, the lower and the upper matrix bounds P_{L1} and P_{U3} represent very satisfactory approximations of the solution matrix *P*, in this case.

Now, consider the same coefficient matrix *A* with singular right-hand side matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues and the trace of the solution *P* are:

$$\lambda_1(P) = 4.07843,\ \lambda_2(P) = 0.00019,\ \lambda_3(P) = 0.461198,$$

$$\lambda_4(P) = 0.293449,\ \lambda_5(P) = 0.23018,\ \lambda_6(P) = 0.0429,\ tr(P) = 6.105423.$$

Matrix *Q* is singular and the available lower bounds for the minimal eigenvalue of *P* are either inapplicable (bounds (1.17), (1.25), (1.69)-(1.75)), or they provide the trivial estimate $\lambda_6(P) \geq 0$ (bounds (1.16), (1.18)). Matrix $\overline{Q} = Q + A^T QA$ is positive definite, with $\lambda_6(\overline{Q}) = 0.1842$, which means that the supposition made in Theorem 2.8 is satisfied. This helps to get the lower eigenvalue bound:

$$\lambda_6(P) = 0.0429 \geq 0.03578 \text{ bound (2.19)}, \quad ¢_L = 16.6\%$$

The upper eigenvalue bound (1.19) requires positive definiteness of Q, and therefore can't be computed, as well. For the solution trace we obtain the estimates:

$tr(P) = 6.105423 \geq 1.12111$, bound (1.20), $\quad ¢_L = 81.64\%$

$$tr(P) = 6.105423 \leq \begin{cases} 6.601, & \text{bound (1.23)} \\ 21.54, & \text{bound (1.24)} \end{cases}, \qquad \begin{matrix} ¢_U = 8.1\% \\ ¢_U = 249.4\% \end{matrix}$$

Obviously, bounds (1.20) and (1.24) are not satisfactory, while the upper trace bound (1.23) is a tight one. The bounds based on the singular value decomposition approach are computed as follows:

$tr(P) = 6.105423 \geq 6.0646$, bound (2.16), $\quad ¢_L = 0.67\%$

$tr(P) = 6.105423 \leq 6.125023$, bound (2.6), $\quad ¢_U = 0.32\%$

$ª_1(P) = 4.07843 \leq 4.276286$, bound (2.17), $\quad ¢_U = 4\%$

Conclusions. The available lower eigenvalue bounds are not computable in this case. The obtained here estimates (2.19) and (2.17) are rather satisfactory. Both the lower (2.16) and the upper (2.6) trace bounds are very tight and are preferable to the respective estimates (1.20) and (1.23).

Example 4.1.3 Distillation column [111].

Consider the stable coefficient matrix

$$A = \begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.306 & -2.132 & 0.9807 & 0 & 0 \\ 0 & 1.595 & -3.149 & 1.547 & 0 \\ 0 & 0.0355 & 2.632 & -4.257 & 1.855 \\ 0 & 0.0023 & 0 & 0.1636 & -0.1625 \end{bmatrix}$$

For $Q = A^T A$, the eigenvalues and the trace of the CALE solution P are computed as follows:

$$ª_1(P) = 3.2518,\ ª_2(P) = 1.7462,\ ª_3(P) = 1.0129,$$

$$ª_4(P) = 0.049,\ ª_5(P) = 0.0068,\ tr(P) = 6.0667.$$

Since $ª_1(A_S) = 0.2746$, the upper bounds based on the condition $A \in \boldsymbol{H}^{-}$ can't be applied, in this case. The following scalar solution bounds were computed:

ª$_5(P) = 0.0068 \geq \begin{cases} 0.00065, & \text{bound } (1.16) \\ 0.005, & \text{bound } (1.17) \\ 0.00065, & \text{bound } (1.18) \\ 0.004, & \text{bound } (1.74) \end{cases}$, ¢$_L = 90.44\%$, ¢$_L = 26.47\%$, ¢$_L = 90.44\%$, ¢$_L = 41.17\%$

$tr(P) = 6.0667 \geq \begin{cases} 4.1313, & \text{bound } (1.20) \\ 2.5804, & \text{bound } (1.21) \\ 5.9381, & \text{bound } (1.22) \\ 5.3969, & \text{bound } (1.74) \end{cases}$, ¢$_L = 31.91\%$, ¢$_L = 57.47\%$, ¢$_L = 2.12\%$, ¢$_L = 11.04\%$

The tightest lower scalar bounds are due to (1.17) and (1.22), i.e.

ª$_5(P) \geq 0.005, \quad tr(P) \geq 5.9381$ (4.18)

Since the orthogonal matrix F is stable, i.e. $A \in \tilde{\boldsymbol{H}}$, all upper bounds based on the singular value decomposition approach are applicable.

By making use of (2.28) and (2.32), an upper and a lower matrix bounds for the CALE solution have been computed:

$P \leq P_{U4} =$ «$_{U4} R_1^{-1}$, «$_{U4} = 0.9688$

$P \geq P_{L1} =$ «$_L X^{-1}$, «$_L = 0.5, \; X = R_2 = (AA^T)^{1/2}$

These bounds were used to compute the improved estimates L_3, U_3 from (2.35) and (2.36), respectively. The following results were obtained:

ª$_5(P) = 0.0068 \geq$ ª$_5(L_3) = 0.0043$, bound (2.41), ¢$_L = 36.76\%$

$tr(P) = 6.0667 \geq tr(L_3) = 6.0371$, bound (2.35), ¢$_L = 0.49\%$

ª$_1(P) = 3.2518 \leq 3.6260$, bound (2.42), ¢$_U = 11.5\%$

$tr(P) = 6.0667 \leq tr(U_3) = 6.4828$, bound (2.43), ¢$_U = 6.85\%$

The lower eigenvalue bound in (4.18) is sharper, in this case. The upper eigenvalue bound (2.42) is satisfactory, while both trace estimates (2.35) and (2.43) are rather tight. As a whole, the obtained bounds are good, which also becomes evident from the indicators in (4.3)-(4.6), computed for the lower matrix bound L_3 and the upper matrix bound U_3 as follows:

$$d = 0, \quad D = 0.3964, \quad t_e(L_3, U_3) = 0.4457$$

$$s_1 = 3.4301, \; s_2 = 1.7577, \; s_3 = 1.0135, \; s_4 = 0.0528, \; s_5 = 0.006; \; t_a(L_2, U_3) = 6.26$$

The “average” solution eigenvalue

$$a_a(P) = \frac{tr(P)}{5} = 1.2133$$

belongs to the tight interval

$$1.2074 = \frac{tr(L_3)}{5} \le a_a(P) \le \frac{tr(U_3)}{5} = 1.2966$$

The eigenvalues of the matrix bounds and the exact solution are presented below:

eigenvalues	L_3	P	U_3
a_1	3.2347	3.2518	3.6260
a_2	1.7435	1.7462	1.7715
a_3	1.0091	1.0129	1.0178
a_4	0.0456	0.0490	0.0599
a_5	0.0043	0.0068	0.0076

Table 4.1 Eigenvalues of the solution *P* and its bounds (Example 4.1.3)

Example 4.1.4

Consider the stable coefficient matrix

$$A = \begin{bmatrix} -120 & -2.2 & 0 & 0.5 & 0 & 0.6 & 0 & 0.7 & 0 & 0.8 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -140 & -2.4 & 0 & 0.5 & 0 & 0.6 & 0 & 0.7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0.5 & -160 & -2.6 & 0 & 0.5 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.6 & 0 & 0.5 & -180 & -2.8 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0.7 & 0 & 0.6 & 0 & 0.5 & -200 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbf{R}_{10}$$

For $Q = I_{10}$, the trace and the eigenvalues of the CALE solution are: $tr(P)$ = 586.2329,

$a_1(P) = 486.7447$, $a_2(P) = 27.9072$, $a_3(P) = 26.0579$, $a_4(P) = 24.0854$, $a_5(P) = 21.4215$,

$a_6(P) = 0.0042$, $a_7(P) = 0.0036$, $a_8(P) = 0.0031$, $a_9(P) = 0.0028$, $a_{10}(P) = 0.0025$.

The symmetric part of *A* is not a negative definite matrix ($\lambda_1(A_S) = 0.0206$) and hence the upper bounds requiring that $A \in \boldsymbol{H}^-$ are not applicable.

(i) Lower bounds:

The lower bounds for the minimal eigenvalue of *P* (1.16)-(1.18), and (1.74) provide the exact value $\lambda_{10}(P) = 0.0025$, and

$$tr(P) = 586.2329 \geq \begin{cases} 0.0250, & \text{bound } (1.20) \\ 0.0063, & \text{bound } (1.21) \\ 0.0625, & \text{bound } (1.22) \\ 578.7848, & \text{bound } (1.74) \end{cases}, \qquad \begin{matrix} \Delta_L = 100\% \\ \Delta_L = 100\% \\ \Delta_L = 100\% \\ \Delta_L = 1.27\% \end{matrix}$$

With the exception of the lower trace bound (1.74), the above estimates are useless.

By means of the singular value decomposition approach the following lower and upper estimates have been obtained.

$$\lambda_{10}(P) = 0.0025 = 0.0025, \text{ bound } (2.18), \qquad \Delta_L = 0.00\%$$

$$tr(P) = 586.2379 \geq 578.7848, \text{ bound } (2.16), \qquad \Delta_L = 1.27\%$$

$$\lambda_1(P) = 486.7447 \leq 486.9776, \text{ bound } (2.17), \qquad \Delta_U = 0.048\%$$

$$tr(P) = 586.2379 \leq 587.6213, \text{ bound } (2.6), \qquad \Delta_U = 0.024\%$$

The trace and the eigenvalue bounds are very tight, in this case. The lower and upper matrix bounds

$$P \geq P_{L1} = \alpha_L X, \quad \alpha_L = 0.5, \; X = R_1^{-1} = (AA^T)^{1/2}$$

$$P \leq P_{U4} = \alpha_{U4} R_1^{-1}, \quad \alpha_{U4} = 0.5076$$

have been obtained from (2.32) and (2.28), respectively. The indicator (4.6) computed for P_{L1}, *P*, and P_{U4}, defines the following sharp interval for the "average" solution eigenvalue $\lambda_a(P)$:

$$57.8785 = \lambda_a(P_{L1}) \leq \lambda_a(P) = 58.6233 \leq \lambda_a(P_{U4}) = 58.7621$$

The next table contains the eigenvalues of the lower matrix bound P_{L1}, of the upper matrix bound, P_{U4} , and the solution matrix *P*:

eigenvalues	P_{L1}	P	P_{U4}
a_1	479.6545	486.7447	486.9776
a_2	27.8875	27.9072	28.31313
a_3	26.0310	26.0579	26.4285
a_4	24.0435	24.0854	24.4124
a_5	21.1503	21.4215	21.4732
a_6	0.0042	0.0042	0.0042
a_7	0.0036	0.0036	0.0036
a_8	0.0031	0.0031	0.0032
a_9	0.0028	0.0028	0.0028
a_{10}	0.0025	0.0025	0.0025

Table 4.2 Eigenvalues of the solution *P* and its bounds (Example 4.1.4)

Obviously, P_{L1} and P_{U4} are very good matrix approximations of the solution matrix *P*.

For more examples illustrating the application of the singular value decomposition approach for the estimation of the CALE solution see [119], where some of the proposed here bounds are used to solve the problem for power system models.

4.2 THE CARE BOUNDS

Example 4.2.1 Consider the matrices $A \in \boldsymbol{R}_6$, $B \in \boldsymbol{R}^6$, $Q \in \boldsymbol{R}_6$, given by:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0.1 & 0 & 0.1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & -1 & 1 & 0 & 0.1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0.1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix *A* is unstable, BB^T and *Q* are singular matrices. Moreover, (*C*, *A*) is not a detectable pair. The CARE (1.6) has a positive semi-definite solution *P* with the following eigenvalues and trace:

$$ª_1(P)=1.88,\ ª_2(P)=1.76,\ ª_3(P)=1.02,$$

$$ª_4(P)=0.68,\ ª_5(P)=0.68\,,\ ª_6(P)=0\,,\ tr(P)=5.1924$$

The available lower and upper eigenvalue and matrix bounds for the CARE solution are valid under the suppositions that BB^T and/or Q are positive definite matrices (see bounds (1.27)-(1.33), (1.36), (1.37), etc.) Since the triple (*A*, *B*, *Q*) is not regular, the lower matrix bounds (2.101), (2.102) can't be applied in this case, as well.

Note, that a satisfying (1.79) scalar ~ does not exist, i.e. the upper estimates (1.80)-(1.82) are also inapplicable, in this case. We shall illustrate the application of the singular value decomposition approach and the based on it Theorem 2.21, for the derivation of various upper bounds for *P*. Consider the matrix in (2.95). Let

$$K=\begin{bmatrix}0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}$$

Then,

$$A_U=A-BK=\begin{bmatrix}-1 & 1 & 0 & 0.1 & 0 & 0.1\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.1 & -1 & 1 & 0 & 0.1\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0.1 & 0 & 0.1 & -1 & 1\\ 0 & 0 & 0 & 0 & -1 & 0\end{bmatrix}$$

is a stable matrix belonging to the set $\tilde{\boldsymbol{H}}$ and therefore, Theorem 2.20 can be applied. The upper matrix bound

$$P_{U1}=«_{U1}R_{1U}\,,\quad «_{U1}=1.2083,$$

for the CARE solution has been computed from (2.99), where $Q+K^TK=I_6$. Its eigenvalues and trace are:

$$ª_1(P_{U1})=2.1159,\ ª_2(P_{U1})=2.1159,\ ª_3(P_{U1})=1.7301,$$

$$ª_4(P_{U1})=0.7667,\ ª_5(P_{U1})=0.7667\,,\ ª_6(P_{U1})=0.7032\,,\ tr(P_{U1})=8.1985$$

i.e.,

$$ª_1(P)=1.88\le ª_1(P_{U1})=2.1159\,,\qquad ¢_U=12.55\% \qquad (4.19)$$

$$tr(P)=5.1924\le tr(P_{U1})=8.1985\,,\qquad ¢_U=57.89\% \qquad (4.20)$$

Since P_{U1} satisfies the matrix inequality (2.113) it can be used to derive tighter bounds in accordance with Theorem 2.24. The upper matrix bounds $U_i, i = 1,2,3$, $M_U = P_{U1}$ have been computed from (2.116), with eigenvalues, traces, and respective estimation errors, as follows:

(i) Bound U_1

$$a_1(U_1) = 1.9686,\ a_2(U_1) = 1.9686,\ a_3(U_1) = 1.7300,$$

$$a_4(U_1) = 0.7106,\ a_5(U_1) = 0.7106,\ a_6(U_1) = 0.7032,\ tr(U_1) = 7.7919$$

$$a_1(P) = 1.88 \le a_1(U_1) = 1.9686, \qquad ¢_U = 4.71\% \tag{4.21}$$

$$tr(P) = 5.1924 \le tr(U_1) = 7.7919, \qquad ¢_U = 50.6\% \tag{4.22}$$

(ii) Bound U_2

$$a_1(U_2) = 1.9011,\ a_2(U_2) = 1.9011,\ a_3(U_2) = 1.7300,$$

$$a_4(U_2) = 0.7032,\ a_5(U_2) = 0.6900,\ a_6(U_2) = 0.6900,\ tr(U_2) = 7.6156$$

$$a_1(P) = 1.88 \le a_1(U_2) = 1.9011, \qquad ¢_U = 1.12\% \tag{4.23}$$

$$tr(P) = 5.1924 \le tr(U_2) = 7.6156, \qquad ¢_U = 46.67\% \tag{4.24}$$

(iii) Bound U_3

$$a_1(U_3) = 1.889,\ a_2(U_3) = 1.889,\ a_3(U_3) = 1.7300,$$

$$a_4(U_3) = 0.7032,\ a_5(U_3) = 0.6838,\ a_6(U_3) = 0.6838,\ tr(U_3) = 7.5555$$

$$a_1(P) = 1.88 \le a_1(U_3) = 1.889, \qquad ¢_U = 0.48\% \tag{4.25}$$

$$tr(P) = 5.1924 \le tr(U_3) = 7.5555, \qquad ¢_U = 45.5\% \tag{4.26}$$

The upper bound for the maximal solution eigenvalue in (4.23) is very tight, while, the upper trace bound (4.24) is satisfactory. This example shows that good upper estimates for the CARE solution matrix are achievable even when BB^T is singular.

Example 4.2.2 Consider the same matrices *A* and *B*, but let

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Q is again positive semi-definite, but now (C, A) is an observable pair. Therefore, Theorem 2.23 can be applied to get lower bounds for the CARE solution P with eigenvalues and trace computed as follows:

$$\lambda_1(P) = 1.8273,\ \lambda_2(P) = 1.3317,\ \lambda_3(P) = 1.2544,$$

$$\lambda_4(P) = 0.6841,\ \lambda_5(P) = 0.4155\ ,\ \lambda_6(P) = 0.1927\ ,\ tr(P) = 5.7058$$

Consider matrix (2.96) and let

$$G = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

which results in:

$$A_L = -A - GQ = \begin{bmatrix} 1 & -3 & 0 & -0.1 & 0 & -10.1 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & -1 & -1 & 0 & 0.1 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & -0.1 & 0 & -0.1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

The above matrix belongs to the set $\tilde{\boldsymbol{H}}$. The lower matrix bound

$$P_{L1} = \frac{1}{\alpha_{L1}} R_{1L}^{-1}, \quad \alpha_{L1} = 8.4779$$

has been obtained from (2.101), where

$$BB^T + GG^T = \begin{bmatrix} 4 & 4 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & -2 & 5 \end{bmatrix}$$

The eigenvalues and the trace of P_{L1} are:

$$a_1(P_{L1}) = 0.4579,\ a_2(P_{L1}) = 0.1181,\ a_3(P_{L1}) = 0.0907,$$

$$a_4(P_{L1}) = 0.0589,\ a_5(P_{L1}) = 0.0512,\ a_6(P_{L1}) = 0.0305,\ tr(P_{L1}) = 0.8072,$$

i.e.,

$$a_6(P) = 0.1927 \geq a_6(P_{L1}) = 0.0305, \qquad ¢_L = 84.17\% \tag{4.27}$$

$$tr(P) = 5.7058 \geq tr(P_{L1}) = 0.8072, \qquad ¢_L = 85.85\% \tag{4.28}$$

The lower matrix bounds $L_i, i = 1,2,3,4$, $M_L = P_{L1}$ have been computed from (2.115), with eigenvalues, traces, and respective estimation errors, as follows:

(i) Bound L_1:

$$a_1(L_1) = 0.8612,\ a_2(L_1) = 0.5143,\ a_3(L_1) = 0.4378,$$

$$a_4(L_1) = 0.2883,\ a_5(L_1) = 0.1619,\ a_6(L_1) = 0.0578,\ tr(L_1) = 2.3220,$$

$$a_6(P) = 0.1927 \geq a_6(L_1) = 0.0578, \qquad ¢_L = 70\% \tag{4.29}$$

$$tr(P) = 5.7058 \geq tr(L_1) = 2.3220, \qquad ¢_L = 59.30\% \tag{4.30}$$

(ii) Bound L_2:

$$a_1(L_2) = 1.0966,\ a_2(L_2) = 0.8924,\ a_3(L_2) = 0.8222,$$

$$a_4(L_2) = 0.3958,\ a_5(L_2) = 0.3477,\ a_6(L_2) = 0.1201,\ tr(L_2) = 3.6749,$$

$$a_6(P) = 0.1927 \geq a_6(L_2) = 0.1201, \qquad ¢_L = 37.68\% \tag{4.31}$$

$$tr(P) = 5.7058 \geq tr(L_2) = 3.6749, \qquad ¢_L = 35.6\% \tag{4.32}$$

(iii) Bound L_3:

$$a_1(L_3) = 1.2178,\ a_2(L_3) = 1.0207,\ a_3(L_3) = 0.9366,$$

$$a_4(L_3) = 0.5176,\ a_5(L_3) = 0.3564,\ a_6(L_3) = 0.1520,\ tr(L_3) = 4.2011,$$

$$\lambda_6(P) = 0.1927 \geq \lambda_6(L_3) = 0.1520, \qquad \delta_L = 21.12\% \tag{4.33}$$

$$tr(P) = 5.7058 \geq tr(L_3) = 4.2011, \qquad \delta_L = 26.37\% \tag{4.34}$$

(iv) Bound L_4:

$$\lambda_1(L_4) = 1.2871,\ \lambda_2(L_4) = 1.0389,\ \lambda_3(L_4) = 0.9513,$$

$$\lambda_4(L_4) = 0.5376,\ \lambda_5(L_4) = 0.3571,\ \lambda_6(L_4) = 0.1783,\ tr(L_4) = 4.3502,$$

$$\lambda_6(P) = 0.1927 \geq \lambda_6(L_4) = 0.1783, \qquad \delta_L = 7.47\% \tag{4.35}$$

$$tr(P) = 5.7058 \geq tr(L_4) = 4.3502, \qquad \delta_L = 23.76\% \tag{4.36}$$

The lower bound (4.35) for the minimal eigenvalue is very sharp, and the lower trace bound (4.36) is rather satisfactory.

Since Q is a singular matrix and $rank(B)$ = 1, the lower bounds (1.27)-(1.29) for the minimal eigenvalue of P are not applicable. Consider the previously defined lower trace bounds:

$$tr(P) = 5.7058 \geq 0.7417, \text{ bound } (1.34), \qquad \delta_L = 87\%$$

$$tr(P) = 5.7058 \geq 0.8724, \text{ bound } (1.35), \qquad \delta_L = 84.71\%$$

Obviously, the trace estimate (4.36) is much tighter then (1.35), in this case.

Example 4.2.3 Consider the same matrices A and B. Let $Q = I_6$. The CARE (1.6) has a positive definite solution P with the following eigenvalues and trace:

$$\lambda_1(P) = 1.8799,\ \lambda_2(P) = 1.7699,\ \lambda_3(P) = 1.2623,$$

$$\lambda_4(P) = 0.6951,\ \lambda_5(P) = 0.6812,\ \lambda_6(P) = 0.4159,\ tr(P) = 6.7043$$

Using the same gain matrix K as in Example 4.2.1, the upper matrix bound

$$P_{U1} = \alpha_{U1} R_{1U}, \quad \alpha_{U1} = 2.2444,$$

for the CARE solution has been computed from (2.99), where

$$Q + K^T K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues and trace of this estimate are:

$$ª_1(P_{U1}) = 3.9302,\ ª_2(P_{U1}) = 3.9302,\ ª_3(P_{U1}) = 3.2136,$$

$$ª_4(P_{U1}) = 1.4240,\ ª_5(P_{U1}) = 1.4240,\ ª_6(P_{U1}) = 1.3062,\ tr(P_{U1}) = 15.2285$$

i.e.,

$$ª_1(P) = 1.8789 \leq ª_1(P_{U1}) = 3.9302, \qquad ¢_U = 109\% \tag{4.37}$$

$$tr(P) = 6.7043 \leq tr(P_{U1}) = 15.2285, \qquad ¢_U = 127.14\% \tag{4.38}$$

Both bounds are not tight, but since P_{U1} satisfies the matrix inequality (2.113), it can be used to derive tighter bounds in accordance with Theorem 2.24. The upper matrix bounds $U_i, i = 1,2,3$, $M_U = P_{U1}$, have been computed from (2.116), with eigenvalues, traces, and respective estimation errors, as follows:

(i) Bound U_1

$$ª_1(U_1) = 2.6766,\ ª_2(U_1) = 2.6765,\ ª_3(U_1) = 2.3498,$$

$$ª_4(U_1) = 1.3498,\ ª_5(U_1) = 0.9316,\ ª_6(U_1) = 0.9115,\ tr(U_1) = 10.8958$$

$$ª_1(P) = 1.8789 \leq ª_1(U_1) = 2.6766, \qquad ¢_U = 42.38\% \tag{4.39}$$

$$tr(P) = 6.7043 \leq tr(U_1) = 10.8958, \qquad ¢_U = 62.52\% \tag{4.40}$$

(ii) Bound U_2

$$ª_1(U_2) = 2.4671,\ ª_2(U_2) = 2.0639,\ ª_3(U_2) = 2.0005,$$

$$ª_4(U_2) = 1.2018,\ ª_5(U_2) = 0.7579,\ ª_6(U_2) = 0.7751,\ tr(U_2) = 9.2664$$

$$ª_1(P) = 1.8789 \leq ª_1(U_2) = 2.4671, \qquad ¢_U = 31.2\% \tag{4.41}$$

$$tr(P) = 6.7043 \leq tr(U_2) = 9.2664, \qquad ¢_U = 38.22\% \tag{4.42}$$

(iii) Bound U_3

$$ª_1(U_3) = 2.3514,\ ª_2(U_3) = 1.958,\ ª_3(U_3) = 1.8737,$$

$$ª_4(U_3) = 1.1930,\ ª_5(U_3) = 0.0.73,\ ª_6(U_3) = 0.7042,\ tr(U_3) = 8.8107$$

$$ª_1(P) = 1.8789 \leq ª_1(U_3) = 2.3514, \qquad ¢_U = 25\% \tag{4.43}$$

$$tr(P) = 6.7043 \leq tr(U_2) = 8.8107, \qquad ¢_U = 31.4\% \tag{4.44}$$

The CARE solution trace bound is evaluated from above as:

$tr(P) \leq 8.1985$, bound (2.110) ¢$_U = 22.3\%$ (4.45)

which is the tightest amongst all similar estimates for this example.

The following lower bounds are computed:

$tr(P) = 6.7043 \geq 0.8750$, bound (1.34) ¢$_L = 87\%$ (4.46)

$tr(P) = 6.7043 \geq 1.025$, bound (1.35) ¢$_L = 84.7\%$ (4.47)

Application of the singular value decomposition approach leads to the following results. Firstly, the matrix in (2.96) is stabilized via the gain matrix $G = 0.5 I_6$, i.e.

$$A_L = -A - 1.5C = \begin{bmatrix} -0.5 & -1 & 0 & -0.1 & 0 & -10.1 \\ 1 & -1.5 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & -0.5 & -1 & 0 & -0.1 \\ 0 & 0 & 1 & -1.5 & 0 & 0 \\ 0 & -0.1 & 0 & -0.1 & -0.5 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1.5 \end{bmatrix}$$

Since $A_L \in \tilde{\boldsymbol{H}}$, the suppositions of Theorem 2.22 are satisfied and the lower matrix bound for the CARE solution

$$P_{L2} = \frac{1}{«_{L2}} R_{2L}, \quad «_{L2} = 1.8386$$

was obtained from (2.102). Its eigenvalues and trace are computed as follows:

$$ª_1(P_{L2}) = 1.0805,\ ª_2(P_{L2}) = 1.0235,\ ª_3(P_{L2}) = 0.9799,$$

$$ª_4(P_{L2}) = 0.5126,\ ª_5(P_{L2}) = 0.4769,\ ª_6(P_{L2}) = 0.2237,\ tr(P_{L2}) = 4.2972,$$

i.e.,

$ª_6(P) = 0.4159 \geq ª_6(P_{L2}) = 0.2237$, ¢$_L = 46.21\%$ (4.48)

$tr(P) = 6.7043 \geq tr(P_{L2}) = 4.2972$, ¢$_L = 35.9\%$ (4.49)

The lower matrix bounds L_i, $i = 1,2,3$, $M_L = P_{L2}$, have been computed from (2.115), with eigenvalues, traces, and respective estimation errors, as follows:

(i) Bound L_1:

$$ª_1(L_1) = 1.1637,\ ª_2(L_1) = 1.1613,\ ª_3(L_1) = 1.0362,$$

$$ª_4(L_1) = 0.5668,\ ª_5(L_1) = 0.4769,\ ª_6(L_1) = 0.2794,\ tr(L_1) = 4.7575,$$

$\lambda_6(P) = 0.4159 \geq \lambda_6(L_1) = 0.2794$, $\qquad \varepsilon_L = 32.82\%$ (4.50)

$tr(P) = 6.7043 \geq tr(L_1) = 4.7575$, $\qquad \varepsilon_L = 29\%$ (4.51)

(ii) Bound L_2:

$$\lambda_1(L_2) = 1.2095,\ \lambda_2(L_2) = 1.1936,\ \lambda_3(L_2) = 1.0431,$$

$$\lambda_4(L_2) = 0.5755,\ \lambda_5(L_2) = 0.5617,\ \lambda_6(L_2) = 0.2877,\ tr(L_2) = 4.871,$$

$\lambda_6(P) = 0.4159 \geq \lambda_6(L_2) = 0.2877$, $\qquad \varepsilon_L = 30.82\%$ (4.52)

$tr(P) = 5.7058 \geq tr(L_2) = 4.871$, $\qquad \varepsilon_L = 27.35\%$ (4.53)

(iii) Bound L_3:

$$\lambda_1(L_3) = 1.2558,\ \lambda_2(L_3) = 1.2464,\ \lambda_3(L_3) = 1.0474,$$

$$\lambda_4(L_3) = 0.5933,\ \lambda_5(L_3) = 0.5738,\ \lambda_6(L_3) = 0.2894,\ tr(L_3) = 5.0061,$$

$\lambda_6(P) = 0.4159 \geq \lambda_6(L_3) = 0.2894$, $\qquad \varepsilon_L = 30.42\%$ (4.54)

$tr(P) = 6.7034 \geq tr(L_3) = 5.0061$, $\qquad \varepsilon_L = 25.33\%$ (4.55)

The obtained lower bounds for the minimal eigenvalue and the trace of the CARE solution (4.54) and (4.55) are much tighter then the similar estimates (4.46) and (4.47), in this case.

Conclusions The considered examples illustrate the ability to derive computable bounds for the CARE solution in cases, when the available estimates are inapplicable, i.e. Q and/or BB^T are singular matrices. All obtained here lower and upper, scalar and matrix bounds are rather satisfactory and some of them represent very tight approximations for the eigenvalues and the trace of the solution matrix. Under some suppositions any matrix bound can be additionally and significantly improved in sense of tightness due to Theorem 2.24 (see the estimates in (4.19)-(4.26), (4.27)-(4.36), (4.37)-(4.44), (4.48)-(4.55)). This influences the tightness of the respective based on it scalar bounds, as well. E.g. the error in the upper bound for the maximal eigenvalue (4.19) is $\varepsilon_U = 12.55\%$, and

the error in the lower bound for the minimal eigenvalue (4.27) is $\varepsilon_L = 84.17\%$. The application of Theorem 2.24 results in new bounds,with $\varepsilon_U = 0.48\%$ (4.25) and $\varepsilon_L = 7.47\%$ (4.35), respectively. The most important contribution concerning the proposed here approach is the extension of the set of admissible matrix triples (A, B, C) for which computable estimates for the CARE solution exist.

4.3 THE DALE BOUNDS

The DALE and the DARE positive (semi)-definite solution matrices can always be bounded from below by the respective right-hand side matrices *Q*. In what follows, more attention will be paid to the upper scalar and matrix bounds.

Example 4.3.1 Consider a stable coefficient matrix in (1.9):

$$A = \begin{bmatrix} 0.52 & -0.36 & 0.21 & -0.73 & 0.45 \\ 0.06 & 0.13 & 0.49 & 0.32 & 0.3 \\ 0.07 & -0.08 & -0.17 & 0.27 & 0.39 \\ -0.47 & 0.02 & 0.14 & -0.05 & 0.22 \\ 0.14 & 0.52 & -0.07 & -0.09 & 0.12 \end{bmatrix}$$

and let

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The trace and the eigenvalues and of the unique positive definite solution of the DALE are: $tr(P) = 3.1332$ and

$$\lambda_1(P) = 1.4006,\ \lambda_2(P) = 1.1031,\ \lambda_3(P) = 0.5451,\ \lambda_4(P) = 0.0773,\ \lambda_5(P) = 0.0071$$

The maximal singular value of the coefficient matrix is $\sigma_1(A) = 1.0934$, i.e. $A \notin \boldsymbol{S^1}$, and none of the available bounds based on this condition are applicable, in this case. The maximal eigenvalue of the matrix product R_1R_2 is computed as $\lambda_1(R_1R_2) = 0.9263$, which means that $A \in \tilde{\boldsymbol{S}}$, in accordance with Lemma 3.1. Therefore, all upper bounds for the DALE solution valid under the supposition that $A \in \tilde{\boldsymbol{S}}$, are computable. Moreover, since *A* is a nonsingular matrix, then according to statement (iv) in Lemma 3.1, R_1^{-1} and R_2 are Lyapunov matrices for *A*. Application of the singular value decomposition approach for the discrete-time case resulted in the following upper bounds for the solution matrix *P*. A matrix estimate

$$P \le P_{U3} = \alpha_{U1} R_1^{-1}, \;\; \alpha_{U1} = 2.2005$$

has been computed using (3.38) and after that improved in accordance with Corollary 3.4. This resulted in the computation of matrices $U_i,\ i = 1,..,10$, satisfying the inequalitites:

$$U_1 \ge U_2 \ge \ldots \ge U_{10} \ge P$$

$$\lambda_1(U_1) \ge \lambda_1(U_2) \ge \ldots \ge \lambda_1(U_{10}) \ge \lambda_1(P)$$

$$tr(U_1) \ge tr(U_2) \ge \ldots \ge tr(U_{10}) \ge tr(P)$$

The scalar bounds improvement and the decrease in the eigenvalue and trace estimation percentage errors, ε_λ and ε_{tr}, respectively, are illustrated by the next table:

Bound	P_{U3}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}
$\lambda_1(\cdot)$	5.06	2.74	2.30	2.08	1.978	1.89	1.796	1.724	1.66	1.61	1.58
$\varepsilon_\lambda(\cdot)$	261	96	64.2	48.5	41.2	35	28.23	23	18.5	15	13
$tr(\cdot)$	18.2	9.367	6.585	5.415	4.913	4.64	4.435	4.272	4.13	4.008	3.9
$\varepsilon_{tr}(\cdot)$	480	199	110	73	57	48	41.5	36.3	32	27.7	24.5

Table 4.3 Bounds for the maximal eigenvalue and the trace (Example 4.3.1)

The above data show that due to Theorem 3.9 and Corollary 3.5 the bounds (3.38) and (3.39) can be significantly improved in sense of tightness. The eigenvalue and trace errors $\varepsilon_\lambda(P_{U3}) = 261\%$, $\varepsilon_{tr}(P_{U3}) = 480\%$ are decreased to $\varepsilon_\lambda(U_{10}) = 13\%$ and $\varepsilon_{tr}(U_{10}) = 24.5\%$, respectively.

The lower eigenvalue bound (1.38) and the lower trace bound (1.41) yield the trivial estimates $\lambda_5(P) \ge 0$ and $tr(P) \ge 0$, in this case. The lower bound (1.40) helps to estimate the solution trace from below as follows:

$tr(P) = 3.1332 \ge \text{bound } (1.40) = 2.4658$, $\qquad \varepsilon_L = 21.3\%$

Using the suggested here bounds we obtain:

$\lambda_5(P) = 0.0071 \ge \text{bound } (3.46) = 0.0030$, $\; k = 4 \qquad \varepsilon_L = 57.75\%$

$tr(P) = 3.1332 \ge \text{bound } (3.20) = 3.1133$, $\; k = 4 \qquad \varepsilon_L = 0.64\%$

Example 4.3.2 L-1011 fighter aircraft [11]

Consider the stable nonsingular coefficient matrix and the positive semi-definite right-hand side matrix in (1.9) given by:

$$A = \begin{bmatrix} 0.9971 & 0.3228 & 0.0825 & -0.4662 \\ -0.0158 & 0.3846 & 0.3414 & -1.4869 \\ 0.0062 & -0.0037 & 0.1159 & 0.5499 \\ 0.0150 & 0.0044 & -0.2178 & 0.7247 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The DALE solution matrix P has the following eigenvalues and trace:

$$\lambda_1(P) = 110.8103,\ \lambda_2(P) = 2.8864,\ \lambda_3(P) = 1.1488,\ \lambda_4(P) = 0.9707,\ tr(P) = 115.8162$$

This example is chosen to illustrate the extension of the validity set for upper DALE bounds. The maximal singular value of A is $\sigma_1(A) = 1.9081$ and the maximal eigenvalue of the matrix product R_1R_2 is computed as $\lambda_1(R_1R_2) = 1.8448$. Therefore, $A \notin \tilde{\boldsymbol{S}}$, and $A \notin \boldsymbol{S}^{\boldsymbol{1}}$ in accordance with Lemma 3.1. In an attempt to determine a Lyapunov matrix for A we shall use Lemma 3.3. The maximal singular values of matrices A^i and the maximal eigenvalues of the matrix products

$$\lambda_1(R_{1i}R_{2i}),\quad R_{1i} = [A^i(A^i)^T]^{1/2},\quad R_{2i} = [(A^i)^T A^i]^{1/2},\ i = 1, 2,..,6$$

are given below:

$A^i, i = 1,..,6$	A	A^2	A^3	A^4	A^5	A^6
$\sigma_1(A^i)$	1.9081	2.2142	2.4029	2.5836	2.6875	2.6996
$\lambda_1(R_{1i}R_{2i})$	1.8448	2.1920	2.1099	1.5467	1.0417	0.7368

Table 4.4 Dependence of $\sigma_1(A^i)$ and $\lambda_1(R_{1i}R_{2i})$ on *i* (Example 4.3.2)

The table shows that as i increases, the maximal singular value $\sigma_1(A^i)$ of $A^i, i = 1,..,6$, increases as well, while $\lambda_1(R_{16}R_{26}) < 1$. This means that $A^6 \in \tilde{\boldsymbol{S}}$, or equivalently, R_{16}^{-1} and

R_{26} are Lyapunov matrices for A^6 (Lemma 3.1, statement (ii)). From Lemma 3.3 follows that

$$X_E = \sum_{i=0}^{5} (A^i)^T R_{16}^{-1} A^i$$

is a Lyapunov matrix for *A*, i.e. $A^T X_E A - X_E < 0$. Using this fact, and having in mind (3.38), an upper matrix bound

$$P \le P_{UE} = \alpha_{UE} X_E\,,\ \alpha_{UE} = \lambda_1[Q(X_E - A^T X_E A)^{-1}] = 10.1159$$

for the DALE solution has been computed. Since this estimate satisfies the matrix inequality

$$A^T P_{UE} A - P_{UE} + Q \le 0$$

it is possible to improve it sense of tightness in accordance with Theorem 3.9. Upper matrix bounds $U_i, i = 1,2,..,20$, for *P* have been computed in accordance with (3.49). The scalar bounds improvement and the decrease in the eigenvalue and trace estimation percentage errors, δ_λ and δ_{tr}, respectively, are illustrated by the next table.

Bound	P_{UE}	U_1	U_5	U_{10}	U_{13}	U_{15}	U_{16}	U_{18}	U_{20}
$\lambda_1(\cdot)$	39743	4499.5	184.77	181.27	153.59	145.77	142.41	136.65	131.9
$\delta_\lambda(\cdot)$	35766	3960	66.74	63.6	38.6	31.55	28.5	23.32	19
$tr(\cdot)$	43830	5624.6	194.34	173.41	158.60	150.8	147.43	141.67	137
$\delta_{tr}(\cdot)$	37744	4756.5	67.8	49.73	36.94	30.21	27.30	22.32	18.3

Table 4.5 Maximal eigenvalue and trace upper bounds (Example 4.3.2)

These results clearly indicate the significant improvement in sense of tightness in the upper bounds for the maximal eigenvalue and the trace of the solution matrix *P*. The eigenvalues of the upper matrix bound U_{20} are:

$$\lambda_1(U_{20}) = 131.9453,\ \lambda_2(U_{20}) = 2.8802,\ \lambda_3(U_{20}) = 1.1514,\ \lambda_4(U_{20}) = 0.9708$$

The solution eigenvalues $\lambda_i(P), i = 2,3,4$ are very tightly estimated from above by the respective eigenvalues $\lambda_i(U_{20}), i = 2,3,4$, since the respective errors in them are computed as follows:

$$\text{¢}_{\text{ª}_2} = 0.13\%,\ \text{¢}_{\text{ª}_3} = 0.23\%,\ \text{¢}_{\text{ª}_4} = 0.01\%$$

The minimal solution eigenvalue and trace can't be estimated by the lower bounds (1.38) and (1.41). The lower trace bound (1.40) provides the estimate

$tr(P) = 115.8162 \geq \text{bound}\,(1.40) = 3.0833,$ $\qquad\qquad$ $\text{¢}_L = 97.33\%$

Using the suggested here lower bounds we obtain the following estimates for the minimal eigenvalue and the trace of the solution matrix:

$\text{ª}_4(P) = 0.9707 \geq \text{bound}\,(3.46) = 0.9692\,,\quad k = 6$ $\qquad\qquad$ $\text{¢}_L = 0.15\%$

$tr(P) = 115.8162 \geq \text{bound}\,(3.20) = 39.8557,$ $\qquad\qquad$ $\text{¢}_L = 65.6\%$

This example clearly indicates that the suggested here bounds for the DALE solution work when the available similar estimates are inapplicable. Moreover, the obtained matrix and scalar approximations are rather sharp.

Example 4.3.3 Consider a stable for all ® coefficient matrix:

$$A = \begin{bmatrix} 0 & \text{®} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{®} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{®} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{®} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{®} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \text{®} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{®} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{®} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{®} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \boldsymbol{R}_{10}$$

i.e. $a_{ij} = \text{®},\ j = i+1,\ a_{ij} = 0,$ otherwise. The maximal singular value of A is equal to ® .

case(i) Let $\text{®} = 0.9$ and

$$Q = \begin{bmatrix} 2 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 1 \\ 1 & 6 & 2 & 2 & 3 & 4 & 4 & 3 & 5 & 4 \\ 1 & 2 & 4 & 2 & 3 & 3 & 3 & 3 & 2 & 3 \\ 0 & 2 & 2 & 3 & 2 & 2 & 3 & 2 & 2 & 2 \\ 2 & 3 & 3 & 2 & 6 & 5 & 4 & 4 & 3 & 3 \\ 2 & 4 & 3 & 2 & 5 & 6 & 4 & 4 & 3 & 4 \\ 1 & 4 & 3 & 3 & 4 & 4 & 6 & 3 & 5 & 4 \\ 2 & 3 & 3 & 2 & 4 & 4 & 3 & 6 & 4 & 4 \\ 1 & 5 & 2 & 2 & 3 & 3 & 5 & 4 & 7 & 5 \\ 1 & 4 & 3 & 2 & 3 & 4 & 4 & 4 & 5 & 6 \end{bmatrix}, \quad a_{10}(Q) = 0.0366$$

The DALE solution matrix P has the following eigenvalues and trace:

$$a_1(P) = 92.132,\ a_2(P) = 12.8678,\ a_3(P) = 12.1099,\ a_4(P) = 9.3498,\ a_5(P) = 9.0068,$$

$$a_6(P) = 7.3498,\ a_7(P) = 6.5344,\ a_8(P) = 5.4579,\ a_9(P) = 3.5990,\ a_{10}(P) = 1.3134,$$

and $tr(P)$ = 160.0946. Since $A \in \tilde{\boldsymbol{S}}$, then $A \in \boldsymbol{S}^{\boldsymbol{1}}$, in accordance with Lemma 3.1, all upper bounds for the DALE solution are applicable in this case.

The application of the singular value decomposition approach leads to the next results.

First of all we need to construct a Lyapunov matrix Φ_1 in (3.4). The singular value decomposition of A (3.1)-(3.3) is:

$$A = U\Sigma V^T, \quad \Sigma = \begin{bmatrix} \Xi_9 & 0_{9,1} \\ 0_{1,9} & 0 \end{bmatrix}, \quad \Xi_9 = 0.9 I_9$$

$$U = I_{10}, \quad V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, the orthogonal matrix defined in (3.3) is $F = V^T U = V^T$, and

$$F_{11}=\begin{bmatrix}0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0\end{bmatrix}\in \boldsymbol{R}_{\boldsymbol{9}},\quad F_{12}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}\in \boldsymbol{R}^{\boldsymbol{9}},\quad F_{21}^T=-\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$$

The inequality in (3.8) is satisfied for all $0<°_1<1.054093$. Let $°_1=1$. This defines the diagonal matrix

$$\Phi_1=\begin{bmatrix}\Xi_9 & 0_{9,1}\\ 0_{1,9} & 1\end{bmatrix},\quad \Xi_9=0.9I_9$$

in (3.4), and Φ_1^{-1} as a Lyapunov matrix for *A*, i.e.

$$A^T\Phi_1^{-1}A-\Phi_1^{-1}\le -0.1I$$

in accordance with Theorem 3.1. Using (3.38) an upper matrix bound

$$P\le P_{U3}=«_{U1}\Phi_1^{-1},\ \ «_{U1}=179.4858$$

has been obtained. Since P_{U3} satisfies the inequality (3.46) it can be improved in sense of tightness via the computation of the upper bounds $U_i,\ i=1,..,10$ from (3.49). The results are given in the next table.

Bound	P_{U3}	U_1	U_3	U_5	U_6	U_7	U_8	U_9	U_{10}
$ª_1(\cdot)$	199.43	194.32	171.41	147	133.81	120.98	109.76	99.987	92.13
$¢_ª(\cdot)$	116.46	110.9	86	59.5	45.24	31.31	19.13	8.53	0
$tr(\cdot)$	19743	1505.8	856.74	491.69	376.63	292.43	232.22	189.73	160.1
$¢_{tr}(\cdot)$	12232	840.6	435.15	207.12	135.25	82.66	45.05	18.51	0

Table 4.6 Maximal eigenvalue and trace upper bounds (Example 4.3.3)

where $¢_{a}$ and $¢_{tr}$ denote the eigenvalue and the trace estimation percentage errors, respectively. This result is illustrated by the following remarkable fact. Substitution of P with U_{10} in the DALE (1.9) leads to:

$$A^T U_{10} A - U_{10} + Q = O = [o_{ij}], \quad \max\left|o_{ij}\right| = 0.17763 \text{ x } 10^{-14}$$

In fact, U_{10} coincides with the solution matrix.

Matrix A is nilpotent with $A^{10} = 0$. Having in mind (1.10) it follows that

$$P = \sum_{i=0}^{9} (A^i)^T Q A^i$$

in this case. We shall illustrate how the bound in (3.25) can be used to get tight upper estimate for the solution matrix. In addition, there exists positive integer p such that $P_{U,p} \equiv P$. Let p = 10, and consider the estimate (3.25):

$$P \le P_{U,10} = Q + ª_1(Q)[\sum_{i=1}^{9} (A^i)^T A^i + \frac{1}{1 - ±_1^2(A)} (A^{10})^T A^{10}] - S(0,9)$$

$$= Q + ª_1(Q) \sum_{i=1}^{9} (A^i)^T A^i - S(0,9)$$

$$= Q + ª_1(Q) \sum_{i=1}^{9} (A^i)^T A^i - \sum_{i=1}^{9} (A^i)^T \Delta_Q A^i$$

$$= Q + \sum_{i=1}^{9} (A^i)^T [ª_1(Q) I - \Delta_Q] A^i$$

$$= Q + \sum_{i=1}^{9} (A^i)^T Q A^i$$

$$= \sum_{i=0}^{9} (A^i)^T Q A^i$$

$$= P$$

The scalar bounds improvement and the decrease in the eigenvalue and trace estimation percentage errors, $¢_{a}$ and $¢_{tr}$, respectively, are illustrated in the Table 4.7, where

$$P_{U,i} = Q + ª_1(Q) \sum_{i=1}^{9} (A^i)^T A^i - S(0,i), \quad i = 1,..,9$$

Bound	$P_{U,1}$	$P_{U,2}$	$P_{U,3}$	$P_{U,4}$	$P_{U,5}$	$P_{U,6}$	$P_{U,7}$	$P_{U,8}$	$P_{U,9}$
ª$_1(\cdot)$	130.24	123.64	117.26	110.66	104.22	98.81	95.21	93.083	92.132
¢$_ª(\cdot)$	41.36	34.20	27.27	20.11	13.12	7.25	3.34	1.03	0
$tr(\cdot)$	607.78	459.26	353.4	279.35	228.84	195.60	175.58	164.77	160.09
¢$_{tr}(\cdot)$	279.64	186.9	120.74	74.5	43	22.18	9.66	2.9	0

Table 4.7 Maximal eigenvalue and trace upper bounds (Example 4.3.3)

The data presented above clearly show the role of the defined in (3.26)-(3.28) term $S(a,q)$ in getting tighter upper estimates for the DALE solution. The respective estimates become sharper as the parameter *i* increases. The exact solution matrix is obtained for *i* = 9. Application of the available upper bounds leads to the following results:

ª$_1(P) = 92.132 \le 174.5826$, bound (1.39), ¢$_U = 89.5\%$

ª$_1(P) = 92.132 \le 174.2023$, bound (1.68), ¢$_U = 89.08\%$

$tr(P) = 160.0946 \le 273.6842$, bound (1.42), ¢$_U = 70.95\%$

$tr(P) = 160.0946 \le 273.6642$, bound (1.67), ¢$_U = 70.94\%$

$tr(P) = 160.0946 \le 1324.7$, bound (1.68), ¢$_U = 727.45\%$

where the scalar estimates due to (1.68) are computed as the maximal eigenvalue and the trace of the upper matrix bound.

4.4 THE DARE BOUNDS

Example 4.4.1 Consider the following unstable coefficient matrix in (1.14):

$$A=\begin{bmatrix} 0 & \sim & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 \\ 0 & -1.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & 0 & -\sim & 0 & 0 & 0.5 & 0 & 0.2 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0.1 & 0 & 0 & 0.35 & 0 & 0 & 0 \end{bmatrix} \in \boldsymbol{R}_{10}$$

and the control matrix of rank 2

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

Let the right-hand side matrix be chosen as:

$$Q=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \geq 0$$

For

$$K=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \boldsymbol{R}_{2,10}$$

the close-loop matrix $A_C = A + BK$ in (3.71) is given by

$$A_C = \begin{bmatrix} 0 & \sim & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & 0 & -\sim & 0 & 0 & 0.5 & 0 & 0.2 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0.1 & 0 & 0 & 0.35 & 0 & 0 & 0 \end{bmatrix}$$

Case (i) Let $\sim = 0.9, \quad = 1$. The positive definite solution of the DARE has the following eigenvalues and trace:

$$\text{a}_1(P) = 2.0001,\ \text{a}_2(P) = 1.8981,\ \text{a}_3(P) = 1.4985,\ \text{a}_4(P) = 1.3178,\ \text{a}_5(P) = 1.0246,$$

$$\text{a}_6(P) = 1.0020,\ \text{a}_7(P) = 1.0001,\ \text{a}_8(P) = 0.7743,\ \text{a}_9(P) = 0.1056,\ \text{a}_{10}(P) = 0.0013,$$

and $tr(P)$ = 10.6403.

Matrix A_C is stable but $\pm_1(A_C) = 1.2272$. According to Lemma 3.5 the solution of the DARE (1.14) is bounded from above by the solution U of the DALE (3.72), where:

$$\overline{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \geq 0$$

Since

$$\text{a}_1(R_1 R_2) < 1, \quad R_1 = (A_C A_C^T)^{1/2}, \quad R_2 = (A_C^T A_C)^{1/2}$$

$A_C \in \tilde{\boldsymbol{S}}$ (see Lemma 3.1, statement (iii)). In other words, the defined in (3.4) and (3.5) matrices Φ_1^{-1}, Φ_2 are Lyapunov matrices for A_C. Since A_C is also nonsingular, then

$\Phi_1 \equiv R_1, \Phi_2 \equiv R_2$, in accordance with Lemma 3.1, statement (iv). The upper matrix bound for the DARE solution P

$$P \le P_{U2} = \alpha_{U2} R_2, \ \ \alpha_{U2} = 11.9859$$

has been obtained using (3.75). This bound can be improved in sense of tightness in accordance with (3.90) since P_{U2} satisfies the inequality (3.88). The scalar bounds improvement and the decrease in the eigenvalue and trace estimation percentage errors, ε_{λ} and ε_{tr} , respectively, are illustrated by the next table.

Bound	P_{U2}	U_{21}	U_{22}	U_{23}	U_{24}	U_{25}
$\lambda_1(\cdot)$	14.7086	11.9601	5.6210	2.3357	2.0479	2.0335
$\varepsilon_{\lambda}(\cdot)$	635.4	497.98	181.04	16.78	2.39	1.67
$tr(\cdot)$	59.7189	25.5318	15.5104	11.8055	11.0301	10.9321
$\varepsilon_{tr}(\cdot)$	461.25	139.95	45.77	10.95	3.66	2.74

Table 4.8 Maximal eigenvalue and trace upper bounds (Example 4.4.1)

Consider the upper matrix bound in (1.59). A new upper matrix bound $\tilde{P}_U$ is obtained for $P_U = U_{25}$. Its eigenvalues and trace are computed as follows:

$$\lambda_1(\tilde{P}_U) = 2.0001,\ \lambda_2(\tilde{P}_U) = 1.9302,\ \lambda_3(\tilde{P}_U) = 1.5562,\ \lambda_4(\tilde{P}_U) = 1.3200,\ \lambda_5(\tilde{P}_U) = 1.026\,,$$

$$\lambda_6(\tilde{P}_U) = 1.0020,\ \lambda_7(\tilde{P}_U) = 1.0001,\ \lambda_8(\tilde{P}_U) = 0.7750,\ \lambda_9(\tilde{P}_U) = 0.1077,\ \lambda_{10}(\tilde{P}_U) = 0.0013\,,$$

and $tr(P)$ = 10.7426. The upper eigenvalue and trace bounds due to $\tilde{P}_U$ are very tight. E.g. $\lambda_1(P) = \lambda_1(\tilde{P}_U)$, $\lambda_{10}(P) = \lambda_{10}(\tilde{P}_U)$ and the trace estimation error is $\varepsilon_{tr}(\tilde{P}_U)$ = 0.96%.

Case (ii) Let ~ $= 0.4$, $\quad = 0.6$ .Then, $\lambda_1(R_1 R_2) = 0.4662$, i.e. $A_C \in \tilde{\boldsymbol{S}}$. The trace and the eigenvalues of the solution matrix P are computed as follows: $tr(P)$ = 8.5901 and

$$\lambda_1(P) = 1.36,\ \lambda_2(P) = 1.3184,\ \lambda_3(P) = 1.1724,\ \lambda_4(P) = 1.0328,\ \lambda_5(P) = 1.0136,$$

$$\lambda_6(P) = 1.0001,\ \lambda_7(P) = 0.9483,\ \lambda_8(P) = 0.5958,\ \lambda_9(P) = 0.1472,\ \lambda_{10}(P) = 0.0014$$

The defined in (3.4) and (3.5) matrices Φ_1^{-1}, Φ_2 are Lyapunov matrices for A_C . Since A_C is also nonsingular, then $\Phi_1 \equiv R_1, \Phi_2 \equiv R_2$, in accordance with Lemma 3.1, statement (iv). The upper matrix bound for the DARE solution P

$$P \le P_{U2} = \alpha_{U2} R_2, \ \alpha_{U2} = 10.8184$$

has been computed from (3.75) and then improved in sense of tightness in accordance with (3.90) since P_{U2} satisfies the inequality (3.88). The scalar bounds improvement and the decrease in the eigenvalue and trace estimation percentage errors, ε_λ and ε_{tr}, respectively, are illustrated by the next table.

Bound	P_{U2}	U_{21}	U_{22}	U_{23}	U_{24}	U_{25}
$\lambda_1(\cdot)$	9.4228	4.3672	1.8812	1.3607	1.3601	1.36
$\varepsilon_\lambda(\cdot)$	592.85	221.12	38.32	0.0515	0.0074	0
$tr(\cdot)$	43.6179	16.0219	10.6974	9.5645	9.1365	9.012
$\varepsilon_{tr}(\cdot)$	407.77	86.52	24.53	11.34	6.36	4.91

Table 4.9 Maximal eigenvalue and trace upper bounds (Example 4.4.1)

A new upper matrix bound $\tilde{P}_U$ is obtained for $P_U = U_{25}$ in (1.59). Its eigenvalues and trace are computed as follows:

$$\lambda_1(\tilde{P}_U) = 1.36,\ \lambda_2(\tilde{P}_U) = 1.32,\ \lambda_3(\tilde{P}_U) = 1.1864,\ \lambda_4(\tilde{P}_U) = 1.0947,\ \lambda_5(\tilde{P}_U) = 1.0281,$$

$$\lambda_6(\tilde{P}_U) = 1.0136,\ \lambda_7(\tilde{P}_U) = 1.0001,\ \lambda_8(\tilde{P}_U) = 0.6069,\ \lambda_9(\tilde{P}_U) = 0.1534,\ \lambda_{10}(\tilde{P}_U) = 0.0015,$$

and $tr(P) = 8.7647$, i.e. the estimated eigenvalues and trace are very close to the exact ones.

REFERENCES

1. Abou-Kandil, H., G. Freiling, V. Ionesco, G. Jacik. Matrix Riccati equations in control and systems theory, Birkhausen Verlag, Basel – Boston, Berlin, 2003.
2. Agathoklis, P. The Lyapunov equation for *n*-dimensional discrete systems. IEEE Trans. on Automat. Contr., v. 35, 448-451, 1988.
3. Anderson, B. A system theory criterion for positive real matrices, SIAM Journ. Contr., v. 5, 171-182, 1967.

4. Anderson, B., J. Moore. Algebraic structure of positive real matrices, SIAM Journ. Contr., v. 6, 615-623, 1968.
5. Anderson, B., P. Agathoklis, E. Jury, M. Mansour. Stability and the matrix Lyapunov equation for discrete 2-dimensional systems. IEEE Trans. on Circuits and Systems, v. 33, 261-266, 1986.
6. Arkun, Y., S. Ramakrishnan. Bounds of the optimal quadratic cost of structure constrained regulators, IEEE Trans. on Automat. Contr., v.28, 924-927,1983.
7. Arnold, W., A. Laub. Generalized eigenproblem algorithms and software for algebraic Riccati equations, Proc. IEEE, v.72, 1746-1754, 1994.
8. Balas, M. Trends in large space structure control theory: fondest hopes, wildest dreams, IEEE Trans. on Automat. Contr., v. 27, 522-535, 1982.
9. Banks, H., K. Kunish. The linear regulation problem for parabolic systems, SIAM. Journ. Contr. and Optimization, v. 22, 684-698, 1984.
10. Bartels, B., G. Stewart. Solution of the equation $AX + XB = C$, Comm. ACM, v.15, 820-826, 1972.
11. Beale, S., B. Shafai. Robust control system design with a proportional-integral controller, Int. J. Contr., v.50, 97-111, 1989.
12. Benner, P. Computational methods for linear quadratic optimization, Supplemento ai Rendiconti del Circolo Matematico di Palermo, series II, v. 58, 21-56, 1999.
13. Benner, P., E. Quintana-Orti. Solving stable generalized Lyapunov equations with the matrix sign function, Numerical Algorithms, v. 20, 75-100, 1999.
14. Benner, P., J. Li, T. Penzl. Numerical solution of large-scale Lyapunov equations, Riccati equations and linear-quadratic optimal control problems, Numeric. Linear Algebra with Appl., v. 15, 755-777, 2008.
15. Bernstein, D., D. Hyland. The optimal projection equation for reduced-order state estimation. IEEE Trans. on Automat. Contr., v. 30, 583-585, 1985.
16. Blondel, V., J. Tsitsiklis. A survey of computational complexity results in systems and control theory, Automatica, v. 36, 1249-1274, 2000.
17. Brockett, R. Finite dimensional linear systems, John Wiley, New York, 1970.

18. Bunse – Gerstner, A., R. Byers, V. Mehrmann. Numerical methods for algebraic Riccati equations. Proc. Workshop on the Riccati Equations in Control Systems and Signals, 107–116, Como, Italy, 1989.
19. Chen, C. Linear system theory and design. Holt, Reinehart and Winston. New York, 1984.
20. Choi, H., T. Kuc. Lower matrix bounds for the continuous algebraic Riccati and Lyapunov equations, Automatica, v. 38, 1147-1152, 2002.
21. Chu, K. The solution of the matrix equations $AXB – CXD = E$ and $(YA - DZ, YC-BZ) = (E, F)$, Linear Algebra Appl., v. 93, 93-105, 1987.
22. Cohn, S., D. Parrish. The behavior of forecast covariances for a Kalman filter in two dimensions, Monthly Weather Review, v. 119, 1757-1785, 1991.
23. Dai, H., Z. Bai. On eigenvalue bounds and iteration methods for discrete algebraic Riccati equations, Journ. of Comput. Math., v. 29. 341-366, 2011.
24. Datta, B. Numerical methods for linear control systems, Elsevier, Academic Press: San Diego, CA, 2004.
25. Davies, R., P. Shi, R. Wiltshire. New lower solution bounds of the continuous algebraic Riccati matrix equation, Linear Algebra and Appl., v. 427, 242-255, 2007.
26. Davies, R., P. Shi, R. Wiltshire. New upper matrix bounds for the solution of the continuous algebraic Riccati matrix equation, Intern. Journ. Contr. Automat. and Syst., v. 6, 776-784, 2008.
27. Davies, R., P. Shi, R. Wiltshire. New upper solution of the discrete algebraic Riccati matrix equations, Journ. of Comput. and Appl. Math., v. 213, 307-315, 2008.
28. De Vligier, J., H. Verbruggen, P. Bruijn. A time-optimal control algorithm for digital computer control, Automatica, v.18, 239-244, 1982.
29. Dou, A. Method of undetermined coefficients in linear differential systems and the matrix equation $YB – AY = F$, SIAM Journ. Appl. Math., v. 14, 691-696, 1966.
30. Fang, Y., K. Loparo, X. Feng. New estimates for solutions of Lyapunov equations, IEEE Trans. on Automat. Contr., v. 42, 408-411, 1997.

31. Fang, Y., K. Loparo, X. Feng. Robust stability and performance analysis: a new approach, Proc. IEEE Conf. on Decision and Control, San Antonio, TX, 1993.
32. Freund, R. Krylov subspaces methods for reduced-order modeling in circuit simulation, Journ. of Comput. and Appl. Math., v.123, 395-421, 2000.
33. Freund, R. Reduced-order modeling techniques based on Krylov subspaces and their use in circuit simulation, Applied and Computational Control, Signals and Circuits, Birkhauser: Boston, MA, 1999.
34. Gajic, Z., M. Quareshi. Lyapunov Matrix equation in system stability and control, Academic Press, 1995.
35. Gardiner, J., A. Laub, J. Amato, C. Moler. Solution of the Sylvester matrix equation $AXB^T + CXD^T = E$, ACM Trans. Math. Software, v. 18, 223-231, 1992.
36. Gardner J., M. Wette, A. Laub, J. Amato, C. Moler. Algorithm 705: A Fortran-77 software package for solving the Sylvester matrix equation $AXB^T + CXD^T = E$, ACM Trans. Math. Software, v. 18, 232-238, 1992.
37. Garloff, J. Bounds for the eigenvalues of the solution of the discrete Riccati and Lyapunov equations and the continuous Lyapunov equation, Intern. Journ. Contr., v. 43, 423-431, 1986.
38. Gawronski, M. Balanced control of flexible structures, Lecture Notes in Contr. and Inform. Sci., v. 211, Springer: Berlin, London, UK, 1996.
39. Geromel, J., J. Bernussou. On bounds of Lyapunov matrix equation, IEEE Trans. on Autom. Contr., v. 24, 482-483, 1979.
40. Gerstner, A. Computational solution of the algebraic Riccati equation, Journ. of Soc. of Instrument and Contr. Eng., v. 35, 632-639, 1996.
41. Golub, G., C. Van Loan. Matrix computations, The John Hopkins University Press, Baltimore, London, 1996.
42. Golub G., S. Nash, C. Van Loan. A Hessenberg-Schur method for the problem $AX + XB = C$, IEEE Trans. on Automat. Contr., v. 24, 909-913, 1979.

43. Haddad, W., D. Bernstein. The optimal projection equations for reduced-order state estimation: the singular measurement noise, IEEE Trans. on Automat. Contr., v. 32, 1135-1143, 1987.
44. Halevi, Y. The optimal reduced-order estimator for systems with singular measurement noise, IEEE Trans. on Automat. Contr., v. 34, 777-781, 1989.
45. Hammarling, S. Numerical solution of the stable non-negative definite Lyapunov equation, IMA Journ. Numer. Anal., 2, 303-323, 1982.
46. Hewer, G., C. Kenney. The sensitivity of the stable Lyapunov equation, SIAM Journ. Contr. Optimization, v. 26, 321-344, 1988.
47. Higham, N. Accuracy and stability of numerical algorithms. SIAM, Philadelphia, PA, 1996.
48. Higham, N., M. Konstantinov, V. Mehrmann, P. Petkov. The sensitivity of computational control problems, IEEE Contr. Systems magazine, v. 24, 28-43, 2004.
49. Horn R., C. Johnson. Matrix analysis, Cambridge University Press, England, 1986.
50. Horn, R. C. Johnson. Topics in matrix analysis. Cambridge University Press, England, 1990.
51. Hyland, D., D. Bernstein. The optimal projection equations for finite-dimensional fixed-order dynamic compensation of infinite-dimensional systems. SIAM Journ. Contr. Optimization, v. 24, 122-151, 1986.
52. Ito, K. Finite-dimensional compensators for infinite-dimensional systems via Galerkin-type approximations, SIAM. Journ. Contr. and Optimization, v. 28, 1251-1269, 1990.
53. Kalman, R. Contributions to the theory of optimal control. Bol. Soc. Mat. Mex., v.5, 102-199, 1960.
54. Kalman, R. Lyapunov functions for the problem of Lurie in automatic control, Proc. Nat. Acad. Sci. USA, v.49, 201-205, 1963.
55. Kalman, R. When is a linear system optimal? Trans. ASME Journ. Basic Eng., ser. D., v.86, 51-60, 1964.

56. Karanam, V. A note on eigenvalue bounds in algebraic Riccati equation, IEEE Trans. on Automat. Contr., v. 28, 109-111, 1983.
57. Khalil, H. Nonlinear systems. MacMillan, New York, 1992.
58. Kim, J., Z. Bien. Some bounds of the solution of the algebraic Riccati equation, IEEE Trans. on Automat. Contr., v. 37, 1209-1210, 1992.
59. Kim, S., P. Park. Matrix bounds of the discrete ARE solution, Systems and Contr. Lett., v. 36, 15-20, 1999.
60. Kim, S., P. Park. Upper bounds of the continuous ARE solution, IEICE Trans. Fundamentals, v. E83-A, 380-385, 2000.
61. Komaroff, N. Diverse bounds for the eigenvalues of the continuous algebraic Riccati equation, IEEE Trans. on Automat. Contr., v. 39, 532-534, 1994.
62. Komaroff, N. Iterative matrix bounds and computational solutions to the discrete algebraic Riccati equation, IEEE Trans. on Automat. Contr., v. 39, 1676-1678, 1994.
63. Komaroff, N. Simultaneous eigenvalue lower bounds for the Lyapunov matrix equation, IEEE Trans. on Automat. Contr., v. 33, 126-128, 1988.
64. Komaroff, N. Upper summation and product bounds for solution eigenvalues of the Lyapunov matrix equation, IEEE Trans. on Automat. Contr., v. 37, 1040-1042, 1992.
65. Komaroff, N. Upper bounds for the eigenvalues of the solution of the discrete Lyapunov matrix equation, IEEE Trans. on Automat. Contr., v. 35, 468-469, 1990.
66. Komaroff, N. Upper bounds for the solution of the discrete Riccati equation, IEEE Trans. on Automat. Contr., v. 37, 1370-1373, 1992.
67. Konstantinov, M., D. Gu, V. Mehrmann, P. Petkov. Perturbation theory for Lyapunov equations, Elsevier, 2003.
68. Kouikoglu, V., Y. Phillis. Trace bounds on the covariances of continuous-time systems with multiplicative noise, IEEE Trans. on Autom. Control., v. 38, 138-142, 1993.
69. Kreisselmeier, G. A solution of the bilinear matrix equation $AY + BY = - Q$, SIAM Journ. Appl. Math., v.23, 334-338, 1972.

70. Kucera, V. A contribution to matrix quadratic equations. IEEE Trans. on Automat. Contr., v. 17, 344-347, 1972.
71. Kucera, V. The discrete Riccati equation of optimal control, Kybernetica, v. 8, 430-447, 1972.
72. Kwon, B., M. Youn, Z. Bien. On bounds of the Riccati and Lyapunov matrix equation. IEEE Trans. on Automat. Contr., v. 30, 1134-1135, 1985.
73. Kwon, W., A. Pearson. A note on the algebraic matrix Riccati equation, IEEE Trans. on Automat. Contr., v. 22, 142-144, 1977.
74. Kwon, W., Y. Moon, S. Ahn. Bounds in algebraic Riccati and Lyapunov equations: a survey and some new results, Intern. Journ. Contr., v. 64, 377-389, 1996.
75. Larin, V., F. Aliev. Generalized Lyapunov equation and factorization of matrix polynomials, Systems and Contr. Lett., v. 21, 485-491, 1993.
76. Lasiecka, I., R. Triggiani. Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory, Lecture Notes in Contr. and Inform. Sci., v.164, Springer-Berlin, 1991.
77. Lasserre J. A trace inequality for matrix product, IEEE Trans. on Automat. Contr., v. 40, 1500-1501, 1995.
78. Laub, A. A Schur method for solving algebraic Riccati equations, IEEE Trans. on Automat. Control, v. 24, 913-921, 1979.
79. Lee, C., C. Chen. Matrix solution bounds of the continuous Lyapunov equation by using a bilinear transformation, Journ. of the Franklin Institute, v. 346, 741-751, 2009.
80. Lee, C. New results for the bounds of the solution for the continuous algebraic matrix Riccati equation, IEEE Trans. on Automat. Contr., v. 42, 118-123, 1997.
81. Lee, C. New upper solution bounds of the continuous algebraic Riccati matrix equation, IEEE Trans. on Automat. Contr., v.51, 330-334, 2006.
82. Lee, C. On the upper and lower bounds of the solution for the continuous matrix Riccati equation, Intern. Journ, Contr., v. 66, 105-118, 1997.

83. Lee, C. Solution bounds of the continuous and discrete Lyapunov matrix equations, J. Optimization Theory Appl., v. 120, 559-578, 2004.
84. Lee C.. Solution bounds of the continuous Riccati matrix equation, IEEE Trans. on Automat. Contr., v. 48, 1409-1413, 2003.
85. Lee, C., T. Lee. On the estimation of solution bounds of the generalized Lyapunov equation and the robust root clustering for the linear perturbed systems, Intern. Journ. Contr., v.74, 996-1008, 2001.
86. Lee, C., T. Li, F. Kung. A new approach for the robust stability of perturbed systems with a class of non-commensurate time delays, IEEE Trans. on Circuits and Systems, v. 40, 605-608, 1993.
87. Lee, C. Upper and lower matrix bounds of the solution of the discrete Lyapunov equation, IEEE Trans. on Automat. Contr., v. 41, 1338-1341, 1996.
88. Lee, D., G. Heo, J. Woo. New bounds using the solution of the discrete Lyapunov matrix equation. Intern. Journ. of Contr., Automat. and Syst., v. 1, 459-463, 2003.
89. Lerer, L., L. Rodman. Sylvester and Lyapunov equations and some interpolation problems for rational matrix functions. Linear Algebra and Appl., v. 185, 83-117, 1993.
90. Lewis, F., K. Ozcaldiran. Geometric structure and feedback in singular systems. IEEE Trans. on Automat. Contr., v. 43, 450-455, 1989.
91. Liu, J., J. Zhang. New solution bounds for the continuous algebraic Riccati equation, Journ. of the Franklin Institute, v. 348, 2128-2141, 2011.
92. Lu, W., H. Wang, A. Antoniou. An efficient method for evaluation of the controllability and observability gramians of 2-D digital filters and systems, IEEE Trans. on Circuits and Syst., v. 39, 695-704, 1992.
93. Ma, E. A finite series solution of the matrix equation $AX - XB = C$, SIAM Journ. Appl. Math., v. 14, 490-495, 1966.
94. Marshall, A., I. Olkin. Inequalities: Theory of majorization and its applications. Academic Press, 1979.
95. Martensson, K. On the matrix Riccati equation, Inform. Sci., v. 3, 17-49, 1971.

96. Mehrmann, V. The autonomous linear-quadratic control problem, theory and numerical solution. Lecture Notes in Contr. and Inform. Sci., No 163, Springer-Verlag, Heidelberg, 1991.
97. Moore, B. Principal component analysis in linear systems: controllability, observability and model reduction, IEEE Trans. on Automat. Contr., v. 26, 17-32, 1981.
98. Mori, T. A note on bounds for the solution of the algebraic matrix Riccati and Lyapunov equations, Trans. of Soc. Instrument. and Contr. Eng., v. 62-A, 760-762, 1979.
99. Mori, T. Another bounds for the solution to the Lyapunov matrix equation, Trans. of Soc. of Instrument. and Contr. Eng., v.63-A, 38-40, 1980.
100. Mori, T., H. Kokame. Stability margin estimation of real Schur polynomials via established stability tests, Trans. IEICE Fundamentals, v. E81-A, 1301-1304, 1998.
101. Mori, T., I. Derese. A brief summary of the bounds on the solution of the algebraic matrix equations in control theory. Intern. Journ. Contr., v. 39, 247-256, 1984.
102. Mori, T., N. Fukuma, M. Kuwahara. Bounds in the Lyapunov matrix differential equation, IEEE Trans. on Automat. Contr., v. 32, 55-57, 1987.
103. Mori, T., N. Fukuma, M. Kuwahara. On the discrete Lyapunov equation. IEEE Trans. on Automat. Contr., v. 27, 463-464, 1982.
104. Mori, T., N. Fukuma, M. Kuwahara. On the discrete Riccati equation, IEEE Trans. on Automat. Contr., v. 32, 828-829, 1987.
105. Newton, G., A. Gould, J. Kaiser. Analytic design of linear feedback control, Wiley, New York, 1957.
106. Ogata, K. Discrete-time control systems. Prentice Hall, Englewood Cliffs, 1987.
107. Papadopoulos, P., A. Laub, G. Ianculescu, J. Ly, C. Kenney, P. Pandey. Optimal control study for the space station solar dynamic power module. Proc. Conf. on Decision and Contr., 2224-2229, Brighton, UK, 1991.

108. Pappas, T., A. Laub, N. Sandell. On the numerical solution of the discrete-time algebraic Riccati equation, IEEE Trans. on Automat. Contr., v. 25, 631-641, 1980.
109. Patel. R., M.Toda. Quantitative measures of robustness for multivariable systems. Proc. Joint American Control Conf., TP8-A, San Francisco, 1980.
110. Penzl, T. Numerical solution of generalized Lyapunov equations, Adv. Comput. Math., v. 8, 33-48, 1998.
111. Petkov, P., N. Christov, M. Konstantinov. A computational algorithm for pole placement assignment of linear multi-input systems, IEEE Trans. on Automat. Contr., v. 31, 1044-1047, 1986.
112. Petkov, P., N. Christov and M. Konstantinov. Computational methods for linear control systems, Prentice-Hall, Hertfordshire, UK, 1991.
113. Petrovic, B., Z. Grujic. The recursive solution of linear quadratic Nash games for weakly coupled interconnected systems, Journ. Optimization Theory and Appl., v. 56, 463-477, 1988.
114. Popov, V. Hyperstability and optimality of automatic systems with several control functions, Rev. Roum. Sci. Tech. ser. Electrotech. Energ., v.9, 629-690, 1964.
115. Potter, J. Matrix quadratic solutions, SIAM Journ. Appl. Math., v. 14, 496-501, 1966.
116. Safonov, M., R. Chiang. A Schur method for balanced truncation model reduction. IEEE Trans. on Automat. Contr., v. 34, 729-733, 1989.
117. Savov, S., I. Popchev. Bounds for the solution of the continuous Lyapunov equation under relaxed validity constraints, Adv. Studies in Contemp. Mathematics, v. 18, 235-243, 2009.
118. Savov, S., I. Popchev. Discrete algebraic Lyapunov equation: lower and upper matrix bounds, Compt. rend. Acad. Bulg. Sci., v. 58, 1159-1162, 2005.
119. Savov, S., I. Popchev. Experimental study of Lyapunov equation solution bounds for power systems models, Cybern. and Inform. Technologies, v.13, 3-12, 2013.

120. Savov, S., I. Popchev. New generalized upper trace bound for the solution of the Lyapunov equation, Journ. of Pure and Appl. Math. , v. 49, 381-389, 2008.
121. Savov, S., I. Popchev. New upper bounds for the CALE: a singular value decomposition approach, Intern. Journ. Contr. and Automat. Syst., v. 6, 1-7, 2008.
122. Savov, S., I. Popchev. New upper estimates for the solution of the continuous algebraic Lyapunov equation, IEEE Trans. Automat. Contr., v. 49, 1841-1842, 2004.
123. Savov, S., I. Popchev. Non-iterative improvement of trace bounds for the Lyapunov equation solution, Compt. rend. Acad. Bulg. Sci., v. 66, 735-738, 2013.
124. Savov, S., I. Popchev. Solution bounds validity for CALE extension via the singular value decomposition approach, Compt. rend. Acad. Bulg. Sci., v. 59, 933-938, 2006.
125. Savov, S., I. Popchev. The continuous algebraic Lyapunov equation; lower and upper matrix bounds, Compt. rend. Acad. Bulg. Sci., v. 58, 1039-1042, 2005.
126. Savov, S., I. Popchev. Valid inner upper matrix bound for the continuous Lyapunov equation, Compt. rend. Acad. Bulg. Sci., v. 59, 1009-1012, 2006.
127. Shapiro, E. On the linear matrix equation, IEEE Trans. on Automat. Contr., v. 19, 594-596, 1974.
128. Shen, X., A. Zelentsovsky. Comments on “Explicit direct solution of the Lyapunov matrix equation”, Journ. of the Franklin Inst., v. 238, 519-522, 1991.
129. Siljak, D. Decentralized control of complex systems. Academic Press, 1991.
130. Sima, V. Algorithms for linear-quadratic optimization, Pure and Appl. Math., v. 200, Marcel Dekker, New York, 1996.
131. Smith, R. Matrix calculations for Lyapunov quadratic forms, Journ. Diff. Equations, v. 2, 208-217, 1966.
132. Stykel, T. Numerical solution and perturbation theory for generalized Lyapunov equations, Linear Algebra and Appl., v. 349, 155-185, 2002.

133. Tippett, M., D. Marchesin. Bounds for the solution of the discrete algebraic Lyapunov equation, IEEE Trans. on Automat. Contr., v.44, 214-218, 1999.
134. Toda, M., R. Patel. Performance bounds for continuous-time filters in the presence of modeling errors, IEEE Trans. on Aerospace and Electronic Syst., v. 14, 912-929, 1978.
135. Wang, S., T. Kuo, C. Hsu. Trace bounds on the solution of the algebraic Riccati and Lyapunov equations, IEEE Trans. on Automat. Contr., v.31, 654-656, 1986.
136. Wang, S., T. Lin. Robust stability of uncertain time delay systems, Intern. Journ. Contr., v.46, 963-976, 1987.
137. Willems, J. Least squares stationary optimal control and the algebraic Riccati equation, IEEE Trans. in Automat. Contr., v. 16, 621-634, 1971.
138. Wonham, W. On a matrix Riccati equation of stochastic control, SIAM Journ. Appl. Math. v. 6, 681-698, 1968.
139. Yasuda, K., K. Hirai. Upper and lower bounds on the solution of the algebraic Riccati equation, IEEE Trans. on Automat. Contr., v. 24, 483-487, 1979.
140. Yaz, E. Bounds for the eigenvalues of the solution matrix of the algebraic Riccati equation, Journ. Syst. Sci., v. 16, 815-820, 1985.
141. Yedavalli, R. Improved measures of stability robustness of linear state space models. IEEE Trans. on Automat. Contr., v. 30, 577-579, 1985.
142. Yedavalli, R. Robust root clustering for linear uncertain systems using generalized Lyapunov theory, Automatica, v. 29, 237-240, 1993.
143. Zhang, J., J. Liu. Matrix bounds for the solution of the continuous algebraic equation, Math. Problems in Eng., v. 2010, Article ID 819064, 2010.