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Approximation of Pure Time Delay Elements by Using Hankel Norm and Balanced Realizations

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Abstract: *The paper considers the problem of pure time delay element approximation by using Hankel norm and balanced realizations. The weighted delay elements are represented in terms of Pade and Kautz series or by using Laguerre, Pade and Kautz shift operator descriptions. Quantitative error bounds on delay approximations are given. Hankel norm and balanced realizations are used to determine the order of truncation. The reduced order models are compared with the full order approximations to determine the efficiency of the presented methods.*

Keywords: *pure time delay, Laguerre, Pade and Kautz shift operator approximation, balanced realization, Hankel norm approximation.*

1. Introduction

Many industrial processes exhibit time delays in their behaviour. Every action of the control input for such processes will affect the measured output after certain amount of time. The delay element models both the existing time delay in transporting energy, materials and information, as well as the existence of higher order terms by accumulation of time lags. The presence of time delays imposes certain limitations on achievable feedback performance. The delay element has a certain destabilizing effect on the dynamic system. This effect can be observed in

Bode diagram: the magnitude response has a constant value, while the phase response has a constant rate of change and increases with frequency to infinity. Time delays increase the system phase lag and also give rise to nonrational transfer functions of the system, making them more difficult to analyze and control. From a mathematical point of view, time delay systems are infinite dimensional, meaning that their state is an infinite dimensional vector. There is a broad range of methods and algorithms for use with rational transfer function models, while relatively few exist for processes with irrational transfer functions, such as those containing time delays.

A natural question arises whether it is posssible to approximate the delay system model by a finite dimensional one like the rational transfer function. The rational function approximation of the time delay element is usually very inaccurate [7, 9, 11]. Any rational transfer function can have only a finite phase lag, whereas the phase lag of the pure time delay is unbounded. The arbitrary large high frequency phase mismatch results in approximation errors of at least hundred percent at frequencies where the phase error is $(2k+1)\pi$ for some natural *k*. Therefore, the rational approximation of the delay makes sense only over a finite frequency band [4, 9, 11]. Potentially better approximations may be produced when properties of some weighting function $G(s)$ is taken into account in the approximation of the delay element, i.e., the weighted delay model $G(s)e^{-\theta s}$ is explored. Weighted approximation is very reasonable since in process industries first order plus time delay and second order plus time delay models are commonly used to describe the system behavior [8].

This paper considers the problem of rational transfer function approximation of pure time delay elements. The time delay is presented by Laguerre, Kautz and Pade power series or shift operator descriptions truncated by using balanced and Hankel norm realizations. Certain frequency weights are added to the time delay element and error bounds are computed for the derived approximations. Several numerical examples are presented, visualizing the accuracy of the time delay approximation.

2. Rational function approximation of pure time delay elements

The main interest for approximating a time delay with some rational function lies in the attempt to deal with a finite dimensional system instead of the corresponding infinite dimensional one. The time delay element is usually approximated by an all-pass rational function. The Hankel singular values for all-pass functions are all unity, therefore the error between an all-pass function and its lower order rational function approximation will be greater or equal to one and hence the approximation will be very inaccurate. This does not imply however when an all-pass system is connected with a rational weighting function.

The approximation methods used most widely in practice are based on the presentation of the delay element as a ratio of two polynomials [7]:

(1) $$e^{-\theta s} \approx \frac{Q_n(-s)}{Q_n(s)},$$

where $Q_n(s)$ is a stable polynomial of degree *n*. The polynomial $Q_n(s)$ when used for the $[n, n]$ Pade series approximation of the delay element is given in the form [7]:

(2) $$Q_n(s) = \sum_{i=0}^{n} \binom{n}{i} \frac{\theta^i (2n-i)!}{(2n)!} s^i = \sum_{i=0}^{n} \frac{\theta^i (2n-i)!\, n!}{(2n)!\,(n-i)!\, i!} s^i .$$

Some low order $Q_n(s)$ polynomials are: $Q_1(s) = 1 + \frac{\theta s}{2}$, $Q_2(s) = 1 + \frac{\theta s}{2} + \frac{\theta^2 s^2}{12}$, $Q_3(s) = 1 + \frac{\theta s}{2} + \frac{\theta^2 s^2}{10} + \frac{\theta^3 s^3}{120}$. If we denote by P_n the $[n, n]$ Pade approximant of the function e^{-s} for $s = j\omega$ we have the following error bound [9]:

(3) $$\left| e^{-j\omega} - P_n(j\omega) \right| \le \begin{cases} 2\left(\frac{|\omega|}{\psi n}\right)^{2n+1}, & |\omega| \le \psi n, \\ 2, & |\omega| \ge \psi n, \end{cases}$$

where $\psi = 2\left(\sqrt{2}/e\right)^{1/2} \approx 1.443$. Pade method gives an optimal convergence rate for rational approximation of functions of the form $H(s) = e^{-\theta s} G(s)$ in terms of H_∞ norm. If the condition $|G(j\omega)| \le M/|\omega|^p$ is satisfied for some $M > 0$ and $p \ge 1$, then for $2n + 1 \ge p$ the following error bound is valid [9]:

(4) $$\left\| e^{-\theta s} G(s) - P_n(s\theta) G(s) \right\|_\infty \le 2M \left(\frac{\theta}{\psi n}\right)^p ,$$

where $\psi = 2\left(\sqrt{2}/e\right)^{1/2} \approx 1.443$.

A new approach for rational approximation of delay systems is based on shift operator techniques [5, 6]. The advantage of using shift operators comes from the fact that many important orthonormal bases such as Laguerre and Kautz bases are known to be induced by the corresponding shift operators. It is also known that the delay operator is a shift operator. A simple approximation technique uses the classical relationship of exponential functions $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$. If the time delay element is presented in the form [3]

(5) $$e^{-\theta s} = \frac{e^{-\theta s/2}}{e^{\theta s/2}} = \lim_{n\to\infty} \frac{(1 - \theta s/2n)^n}{(1 + \theta s/2n)^n},$$

the shift operator S_{L}^n defined by

(6) $$S_{\mathrm{L}}^n f = \left(\frac{1 - \theta s/2n}{1 + \theta s/2n}\right)^n f, \quad f \in H_2 ,$$

is called a multiple Laguerre shift with multiplicity *n*. The error between the time delay and the Laguerre shift is bounded by the expression [3]

(7) $$\left|e^{-j\omega\theta}-S_{\mathrm{L}}^{n}\right|\le\min\left(2,\frac{n}{12}\phi_{n}^{3}(\omega\theta)\right),$$

where $\phi_n(\omega\theta)=\dfrac{\omega\theta}{n}$. If a stable delay system is presented by the relation $H(s)=e^{-\theta s}G(s)$, where the rational function $G(s)$ has a relative degree k and continuous upper bound along the imaginary axis given by

(8) $$|G(j\omega)|\le\begin{cases}M_{\mathrm{l}}/\omega^{m}, & \omega\le\omega_{\mathrm{c}}\\ M_{\mathrm{h}}/\omega^{k}, & \omega\ge\omega_{\mathrm{c}}\end{cases},$$

where low and high conctants M_{l}, $M_{\mathrm{h}}\ge 0$, $\omega_{\mathrm{c}}>0$ are such that $M_{\mathrm{l}}\omega_{\mathrm{c}}^{k}=M_{\mathrm{h}}\omega_{\mathrm{c}}^{m}$, $m\le 3$, then the error between the Laguerre shift approximation and the delay system is given by the formula as [3]:

(9) $$\text{if }\ \omega_{\mathrm{c}}\theta\ge\beta n^{2/3}\ \text{ then }\ \left\|H-S_{\mathrm{L}}^{n}G\right\|_{\infty}\le\begin{cases}2M_{\mathrm{l}}\left(\dfrac{\theta}{\beta n^{2/3}}\right), & m\ge 0,\\ 2\dfrac{M_{\mathrm{l}}}{\omega_{\mathrm{c}}^{m}}=2\dfrac{M_{\mathrm{h}}}{\omega_{\mathrm{c}}^{k}}, & m\le 0;\end{cases}$$

(10) $$\text{if }\omega_{\mathrm{c}}\theta\le\beta n^{2/3},$$

$$\left\|H-S_{\mathrm{L}}^{n}G\right\|_{\infty}\le\begin{cases}2M_{\mathrm{l}}\omega_{\mathrm{c}}^{3-m}\left(\dfrac{\theta}{\beta n^{2/3}}\right)^{3}=2M_{\mathrm{h}}\omega_{\mathrm{c}}^{3-k}\left(\dfrac{\theta}{\beta n^{2/3}}\right)^{3}, & k\ge 3,\\ 2M_{\mathrm{h}}\left(\dfrac{\theta}{\beta n^{2/3}}\right)^{k}, & k\le 3,\end{cases}$$

where $\beta=2\left(3^{1/3}\right)$.

Another important shift operator is the Kautz shift $S_{\mathrm{K}}^{n}:H_2\to H_2$ defined by [5]

(11) $$S_{\mathrm{K}}^{n}f=\left[\frac{1-\dfrac{\theta s}{2n}+\dfrac{1}{2}\left(\dfrac{\theta s}{2n}\right)^{2}}{1+\dfrac{\theta s}{2n}+\dfrac{1}{2}\left(\dfrac{\theta s}{2n}\right)^{2}}\right]^{n}f,\quad f\in H_2,$$

and called a multiple Kautz shift of multiplicity *2n*. The error bound between the Kautz shift and the delay system is given by the inequality [5]

(12) $$\left\|H-S_{\mathrm{K}}^{n}G\right\|_{\infty}\le 2\frac{(C\theta)^{m}}{48^{1/3}}\|G\|_{\infty}\frac{1}{n^{2m/3}},$$

where C is a constant defined by the expressions $|G(j\omega)|\le\|G\|_{\infty}\dfrac{C^{m}}{\omega^{m}}$, $C<\dfrac{48^{1/3}n^{2/3}}{\theta}$ and $m=1,2$ is the relative degree of $G(s)$. If $C\ge\dfrac{48^{1/3}n^{2/3}}{\theta}$, the error bound is presented as

(13) $$\left\| H - S_{\mathrm{K}}^{n} G \right\|_{\infty} \le \frac{(C\theta)^{3}}{24} \|G\|_{\infty} \frac{1}{n^{2}} .$$

In the case when $m \ge 3$, the error bound can be presented as follows [5]:

(14) $$\left\| H - S_{\mathrm{K}}^{n} G \right\|_{\infty} \le \frac{\theta^{3}}{24} \lim_{\beta \to \infty} \varphi(\beta) \frac{1}{n^{2}} ,$$

where $\varphi(\beta) = \sup\limits_{0 \le \omega \le \beta} \omega^{3} \left| G(j\omega) \right|$ and $n \ge \left[\frac{48(C\theta)^{m-3}}{\left(2\sqrt{2}\right)^{m}} \right]^{1/(m-2)}$. An extension of Kautz shift is the multiple Pade shift of multiplicity 2*n*, $S_{\mathrm{P}}^{n} : H_{2} \to H_{2}$ also known as Pade-2 shift and defined by the expression [6]

(15) $$S_{\mathrm{P}}^{n} f = \left[\frac{1 - \frac{\theta s}{2n} + \frac{1}{3}\left(\frac{\theta s}{2n} \right)^{2}}{1 + \frac{\theta s}{2n} + \frac{1}{3}\left(\frac{\theta s}{2n} \right)^{2}} \right]^{n} f , \quad f \in H_{2} .$$

The error bound between the multiple Pade-2 shift and the delay system is given by the inequality [6]

(16) $$\left\| H - S_{\mathrm{P}}^{n} G \right\|_{\infty} \le 2 \left(\frac{C\theta}{1440^{1/5}} \right)^{m} \|G\|_{\infty} \frac{1}{n^{4m/5}} ,$$

where $m = 1, 2, 3, 4$ and C is a constant satisfying $\left| G(j\omega) \right| \le \|G\|_{\infty} \frac{C^{m}}{\omega^{m}}$ and also $C < 1440^{1/5} \frac{n^{4/5}}{\theta}$. For $m \ge 5$ if $n \ge \left[\frac{1440(C\theta)^{m-5}}{\left(2\sqrt{3}\right)^{m}} \right]^{1/(m-4)}$ and for $m = 1, 2, 3, 4$ if $C \ge 1440^{1/5} \frac{n^{4/5}}{\theta}$, the following error bound is valid:

(17) $$\left\| H - S_{\mathrm{P}}^{n} G \right\|_{\infty} \le \frac{(C\theta)^{5}}{720} \|G\|_{\infty} \frac{1}{n^{4}} .$$

The case when the relative degree is $m \ge 5$, the error bound can be simplified as follows:

(18) $$\left\| H - S_{\mathrm{P}}^{n} G \right\|_{\infty} \le \frac{\theta^{5}}{720} \lim_{\beta \to \infty} \varphi_{5}(\beta) \frac{1}{n^{4}} , \quad n \ge 3 ,$$

where $\varphi_{k}(\beta) = \sup\limits_{0 \le \omega \le \beta} \omega^{k} \left| G(j\omega) \right|$, $\beta \ge 0$.

The pure time delay element is modeled by using series or shift operator approximation. The order of truncation of series representation or the multiplicity number of shift operator description are determined by applying the Hankel norm and balanced realizations. These quantities are determined by exploring the approximation errors of the corresponding realizations.

3. Balanced truncation and Hankel norm approximation of delay elements

A popular method for model reduction of finite dimensional systems which can be used in some infinite dimensional cases is the balanced relization method. A balanced realization is the one where the controllability and observability gramians are equal and diagonal. The diagonal entries of the gramians are called Hankel singular values. By truncating the state space matrices of a balanced realization we obtain a reduced order model with good approximation properties. Model reduction requires the elimination of some of the state variables from the original system representation. The system is first transformed into a balanced form and then some of the state variables are truncated while preserving stability.

Assume a stable linear time invariant system described by its state space model:

(19) $$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \qquad y(t) = Cx(t), \quad x(0) = x_0 .$$

Consider the controllability operator of the system defined as:

(20) $$L_c : L_2(0,\infty) \to R^n, \text{ where } (L_c u)(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

and the observability operator of the system defined as

(21) $$L_o : R^n \to L_2(0,\infty), \text{ where } (L_o x_0)(t) = Ce^{At} x_0 ,$$

then the controllability and observability gramians

(22) $$W_c(0,t) = \int_0^t e^{A\tau} BB^* e^{A^*\tau} d\tau \quad \text{and} \quad W_o(0,t) = \int_0^t e^{A^*\tau} C^* C e^{A\tau} d\tau$$

are the matrix representations of the maps $L_c L_c^*$ and $L_o^* L_o$, where with the star superscript the adjoint operator is assigned. If the linear system is stable, controllable and observable, then the gramians $W_c = \lim_{t\to\infty} W_c(0,t)$ and $W_o = \lim_{t\to\infty} W_o(0,t)$ are the unique solutions of Lyapunov equations:

(23) $$AW_c + W_c A^* + BB^* = 0 \quad \text{and} \quad A^* W_o + W_o A + C^* C = 0 .$$

The smallest amount of energy needed to move the system from zero to state x is given by $E_c = x^* W_c^{-1} x$, while the energy obtained by observing the output of the system with an initial condition x and no input function is given by $E_o = x^* W_o x$. Therefore, one way to reduce the number of states is to eliminate those which require a large amount of input energy E_c to be reached and yield a small amount of observation energy E_o at the output. The goal is to look for a basis in the state space where controllability and observability are equivalent in some sense. Such a basis exists if $W_c = W_o = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, σ_i, $i = 1, 2, \ldots, n$, are the Hankel singular values of the system. Approximation in this basis takes place by truncating the initial state vector $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ to the reduced state vector $\tilde{x} = [x_1, x_2, \ldots, x_k]^{\mathrm{T}}$ for $k < n$. Approximation by balanced truncation preserves stability and the H_∞ norm of the error between the original and the truncated system is given by the expression

(24) $$\left\| \Sigma - \tilde{\Sigma} \right\|_\infty \le 2(\sigma_{k+1} + \ldots + \sigma_n).$$

A balancing transformation is computed as $P = S^{-1/2} U^* R$, where $W_{\mathrm{o}} = R^* R$ is a Cholesky decomposition of the observability gramian and $R W_{\mathrm{c}} R^* = U S^2 U^*$ is the singular value decomposition of the expression in the left side. This type of balancing is known as Lyapunov balancing because it is based on solving Lyapunov equations for the gramians. This approach may turn out to be inefficient, especially for large problems due to the rapid decay of the Hankel singular values. To avoid these difficulties some modifications of the original balancing algorithm are proposed to prevent matrix inversion. One of the most popular modification of the balancing algorithm is the Square Root Algorithm [1, 10]:

- If we partition the matrices $W = [W_1 \quad W_2]$ and $V = [V_1 \quad V_2]$ and $W_{\mathrm{o}} = LL^*$ (Cholesky decomposition)
- $W_{\mathrm{c}} = UU^*$ (Cholesky decomposition)
- $U^* L = W \Sigma V^*$ (SVD decomposition)
- $P = \Sigma^{-1/2} V^* L^*$ (similarity transformation)
- $P^{-1} = U W \Sigma^{-1/2}$ (similarity transformation)

$$d\,\Sigma = \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix},$$

we obtain the similarity transformations as follows: $P_1 = \Sigma_1^{-1/2} V_1^* L^* \in R^{k \times n}$ and $P_1^{-1} = U W_1 \Sigma_1^{-1/2} \in R^{n \times k}$. The reduced order system of dimension k obtained from the original one by balanced truncation is

$$\Sigma_1 = \left(\begin{array}{c|c} P_1 A P_1^{-1} & P_1 B \\ \hline C P_1^{-1} & \end{array} \right).$$

Finally, we obtain the error between the system described by $H(s) = e^{-\theta s} G(s)$ with $G(s)$ stable, rational weighting function and its balanced truncation of the delay series approximation. To obtain the error bound we use the triangle property of norms:

(25) $$\left\| H - \tilde{\Sigma}_k \right\|_\infty = \left\| H - S_\bullet^n G + S_\bullet^n G - \tilde{\Sigma}_k \right\|_\infty \le \left\| H - S_\bullet^n G \right\|_\infty + \left\| S_\bullet^n G - \tilde{\Sigma}_k \right\|_\infty,$$

where $S_\bullet^n$ denotes any of the delay element shift approximation presented in Section 2. The first error norm in (25) is bounded by the inequalities (9), (10), (12), (13), (14) or (16), (17), (18) and the second error norm is bounded by the term (24).

The Hankel norm approximation is based on the norm induced by the Hankel operator. In time domain, the Hankel operator is defined as

$\Gamma_H : L_2(-\infty, 0) \to L_2(0, \infty)$, where $\Gamma_H u = P_+(h \times u)$ is a projection on the positive time axis of the signal $u \in L_2(-\infty, 0)$ convolved with the impulse response $h(\cdot)$. The result of this convolution is obtained as

(26) $$(\Gamma_H u)(t) = \begin{cases} \int_{-\infty}^{0} h(t-\tau)u(\tau)d\tau, & t \ge 0, \\ 0, & t < 0. \end{cases}$$

The Hankel operator has the interpretation of the system future output $y(t) = \Gamma_H u(t)$, $t \ge 0$, based on the past input $u(t)$, $t \le 0$. The Hankel operator can also be presented as a composition of maps from the past input to the initial state and from the initial state to the future output. This composition of maps is presented by two other operators [12]: the controllability operator $L_c : L_2(-\infty, 0) \to R^n$, where $L_c u = \int_{-\infty}^{0} e^{-A\tau} Bu(\tau) d\tau$ and the observability operator $L_o : R^n \to L_2(0, \infty)$, where $L_o x_0 = Ce^{At} x_0$, $t \ge 0$. Thus, the Hankel operator can be considered as a composition of the controllability and observability maps, i.e., $\Gamma_H = L_o \circ L_c = L_o L_c$. The 2-norm of a stable system Hankel operator is given as

(27) $$\|\Gamma_H\|_2 = \|\Gamma_H^* \Gamma_H\|_2^{1/2} = \sqrt{\rho(W_c W_o)},$$

where $\rho(W_c W_o)$ presents the spectral radius of the gramians product and is known as the Hankel norm of the system Σ denoted by $\|\Sigma\|_H$. The Hankel norm is induced 2-norm from the past inputs to future outputs. The Hankel norm approximation theory is based on the following results [1]: *i*) given stable systems Σ and $\tilde{\Sigma}_k$ of degrees *n* and *k*, where $n > k$, there holds $\|\Sigma - \tilde{\Sigma}_k\|_H \ge \sigma_{k+1}(\Sigma)$; *ii*) the two-norm of any L_2 system Σ is no less than the Hankel norm of its stable part Σ_+, i.e., $\|\Sigma\|_2 \ge \|\Sigma_+\|_H$; *iii*) given a stable system Σ, there exists a system $\tilde{\Sigma}_k$, having exactly *k* stable poles and $\|\Sigma - \tilde{\Sigma}_k\|_2 = \sigma_{k+1}(\Sigma)$. Furthermore, $\Sigma - \tilde{\Sigma}_k$ is all-pass. The system $\tilde{\Sigma}_k$ is called all-pass dilation of the system Σ; *iv*) given $k \in \{0, 1, \ldots, n-1\}$, a stable system Σ and a positive number ε, such that $\sigma_k(\Sigma) > \varepsilon > \sigma_{k+1}(\Sigma)$, there exists a system $\tilde{\Sigma}_k$ with *k* stable poles and $\|\Sigma - \tilde{\Sigma}_k\|_2 = \varepsilon$. Furthermore, $\Sigma - \tilde{\Sigma}_k$ is all-pass and $\tilde{\Sigma}_k$ is called ε all-pass dilation of Σ. If $\tilde{\Sigma}_k$ is ε all-pass dilation of Σ and $\sigma_{k+1}(\Sigma) \le \varepsilon \le \sigma_k(\Sigma)$, then $\tilde{\Sigma}_k$ has exactly *k* stable poles and $\sigma_{k+1}(\Sigma) \le \|\Sigma - \tilde{\Sigma}_k\|_H < \varepsilon$. If $\sigma_{k+1}(\Sigma) = \varepsilon$, then $\|\Sigma - \tilde{\Sigma}_k\|_H = \sigma_{k+1}(\Sigma)$.

In state space, the optimal Hankel norm approximation is obtained as follows [2]. Assume that σ is a Hankel singular value of multiplicity *r* of the system Σ. Transform the system into a balanced form and partition the system matrices as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix},\quad D,\quad W_{\mathrm{c}} = \begin{pmatrix} W_{\mathrm{c},11} & 0 \\ 0 & \sigma I_{\mathrm{r}} \end{pmatrix},$$

$$W_{\mathrm{o}} = \begin{pmatrix} W_{\mathrm{o},11} & 0 \\ 0 & \sigma I_{\mathrm{r}} \end{pmatrix}.$$

Then the approximation problem solution is given by the system matrices:

- $\hat{A} = \Gamma^{-1}\left(\sigma^2 A_{11}^* + W_{\mathrm{o},11} A_{11} W_{\mathrm{c},11} - \sigma C_1^* U B_1^*\right)$,
- $\hat{B} = \Gamma^{-1}\left(W_{\mathrm{o},11} B_1 + \sigma C_1^* U\right)$,
- $\hat{C} = C_1 W_{\mathrm{c},11} + \sigma U B_1^*$,
- $\hat{D} = D - \sigma U$,
- $\Gamma = W_{\mathrm{c},11} W_{\mathrm{o},11} - \sigma^2 I$,

where U is a unitary matrix satisfying $B_2 = -C_2^* U$, $U^* U = I$ and σ is the Hankel singular value determining the size of the error.

4. Experimental results

We examine a pure time delay element with a stable rational weighting function of first and second order. This type of transfer functions is quite popular in practice because it fits the models of a large variety of industrial control processes. Consider the first order model with a time delay $H(s) = \dfrac{e^{-\theta s}}{Ts+1}$. Assume that $\theta = 1\,\mathrm{s}$ and T takes the values 0, 0.1, 1 and 10 s. We consider first the $[4, 4]$ Pade approximation of the delay element. The coefficients of the approximating polynomial are obtained as follows:

$$Q_n = [0.0006 \quad 0.0119 \quad 0.1071 \quad 0.5 \quad 1.0].$$

The Hankel singular values of the rational approximation of $H(s)$, where the delay element is approximated by $[4, 4]$ Pade series, are shown in Fig. 1.

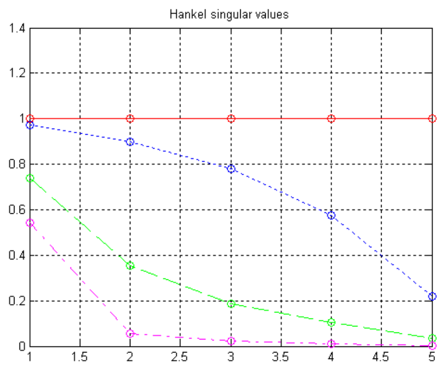

Fig. 1. Hankel singular values of [4, 4] Pade approximation with 1st order weight: T=0 (---); T = 0.1 (…); T = 1 (- -); T = 10 (-.-)

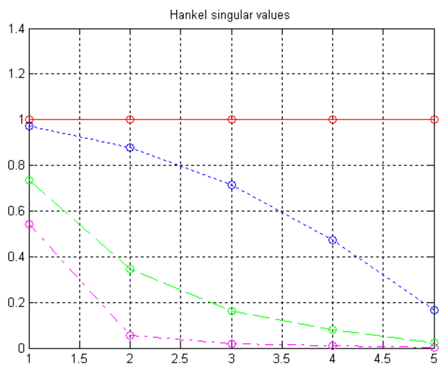

Fig. 2. Hankel singular values of 4th order Laguerre shift 1st order weight: T = 0 (---); T = 0.1 (…); T = 1 (- -); T = 10 (-.-)

The first observation from Fig. 1 is that when $T = 0$ or the approximation is for the pure delay element without a weighting function, then all Hankel singular values of the rational function are equal to one and the error of approximation cannot be reduced further. This result shows that it is not possible to approximate the delay operator arbitrarily closely by rational functions, but the situation changes when the approximation method is applied to weighted delay elements. The second observation is that the error of approximation decreases by increasing the time constant of the weighting function. The Hankel singular values of a multiple Laguerre shift with multiplicity *n* approximation for the same values of *T* are shown in Fig. 2. The approximation polynomial has the following coefficients:

$$Q_n = [0.0002 \quad 0.0078 \quad 0.0938 \quad 0.5 \quad 1.0].$$

Insignificant difference for the Hankel singular values in the case of the fourth order Pade approximation is observed. For example, for $T = 1\,\mathrm{s}$ the $[4, 4]$ Pade series approximation has Hankel singular values $S = \{0.7373, 0.3528, 0.1846, 0.1054, 0.0363\}$, while the Laguerre shift 4 approximation has $S = \{0.7367, 0.3445, 0.1622, 0.0781, 0.0237\}$.

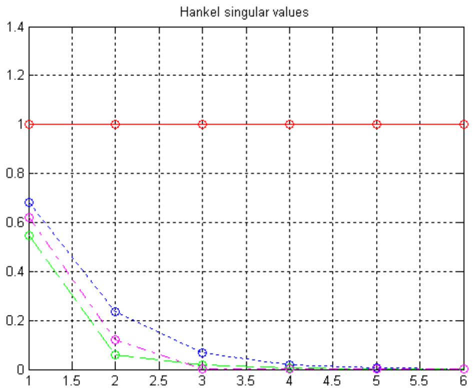

Fig. 3. Hankel singular values of 4th order Laguerre shift: T_1=T_2=0 (---); T_1=0.5, T_2=2 (…); T_1=0.1, T_2=10 (- -); T_1=10, T_2=10 (-.-.)

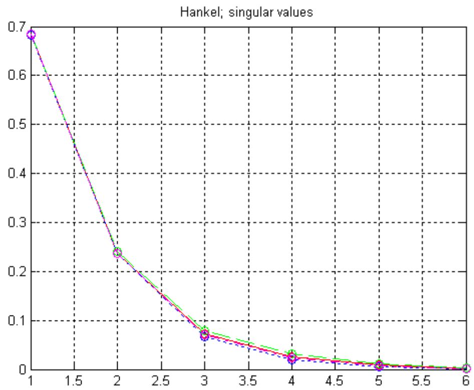

Fig. 4. Hankel singular values of 4th order: Pade (---); shift Laguerre-4 (…); shift Kautz-2 (- -); shift Pade-2 (-.-.)

It is seen again that by slowing down the filter dynamics, the approximation error is reduced. Next, we consider a time delay element with a second order weighting function of the form $H(s) = \dfrac{e^{-\theta s}}{(T_1 s + 1)(T_2 s + 1)}$. We explore four cases for different filter time constants: $T_1 = 0$, $T_2 = 0$; $T_1 = 0.5\,\mathrm{s}$, $T_2 = 2\,\mathrm{s}$; $T_1 = 0.1\,\mathrm{s}$, $T_2 = 10\,\mathrm{s}$ and $T_1 = 1\,\mathrm{s}$, $T_2 = 10\,\mathrm{s}$. The approximation model used is Laguerre shift with multiplicity $n = 4$. The results are shown in Fig. 3. Similarly to the case of the first order weighting function when $T_1 = T_2 = 0$, the Hankel singular values are equal to one. When one of the time constants is increasing, the Hankel singular values have higher rate of decline and the approximation error becomes smaller.

When both time constants are large, the decline rate is the fastest one. Fig. 4 shows the Hankel singular values for $T_1 = 0.5$ s and $T_2 = 2$ s for the approximation models: $[4, 4]$ Pade series, Laguerre shift with n = 4, Pade and Kautz shift with n = 2. It is observed that all four approximation models give almost the same Hankel singular values with the same rate of decay. This result shows that in the low pass frequency range the approximation methods have similar approximation capabilities.

We consider a model of time delay with a first order weight: $H(s) = \dfrac{e^{-\theta s}}{Ts+1}$. Assume that $\theta = 5$ s and $T = 10$ s. The Hankel singular values for $[4, 4]$ Pade series approximation are $S = \{0.6559, 0.2186, \cdots 0.0986, 0.0547, 0.0188\}$ and for shift Laguerre-4 approximation are $S = \{0.6557, 0.2114, 0.0869, 0.0404, 0.0121\}$. For model reduction, we apply the balanced truncation method. Fig. 5 presents the unit step responses for a weighted time delay element which is approximated by $[4, 4]$ Pade series. The time responses for the full order system, the reduced fourth, third, second and first order systems are shown. For all approximation models, except for the first order approximation, the difference in the time response appears only in the steady state value. Similar results are observed for the shift Laguerre-4 approximation models in Fig. 6. It is observed that the reduced fourth and third order models closely approach the full order model. Therefore, the balanced truncation method can reliably be used for weighted approximation of time delay elements. Next, we assume a time delay element with a second order weighting function $H(s) = \dfrac{e^{-\theta s}}{(T_1 s+1)(T_2 s+1)}$, where $T_1 = 5$ s, $T_2 = 0.2$ s and $\theta = 3$ s.

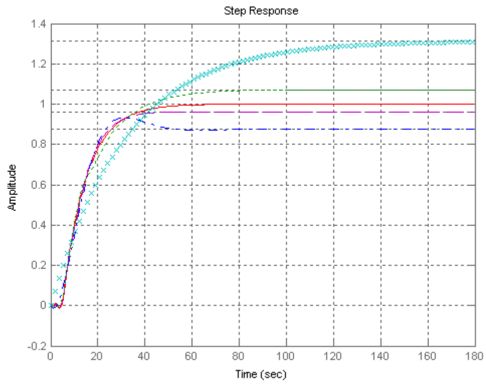

Fig. 5. Unit step response of $[4, 4]$ Pade approximation with 1st order weight: full order (---); reduced 4th order (- -); reduced 3rd order (…); reduced 2nd order (-˙-˙); reduced 1st order (xxx)

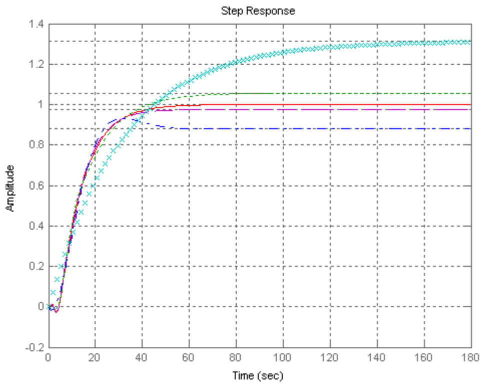

Fig. 6. Unit step response of shift Laguerre-4 approximation 1st order weight: full order (---); reduced 4th order (- -); reduced 3rd order (…); reduced 2nd order (-,-); reduced 1st order (xxx)

We consider the model of multiple Laguerre shift with multiplicity $n = 4$ approximation. The Hankel singular values of the weighted delay element $H(s)$ approximated by the shift Laguerre-4 are: $S = \{0.6821, 0.2543, 0.1062, 0.0483, 0.0169, 0.0026\}$. The step responses to the full order model, the reduced fifth, fourth and third order models by appplying the Hankel norm approximation technique for model reduction are shown on Fig. 7.

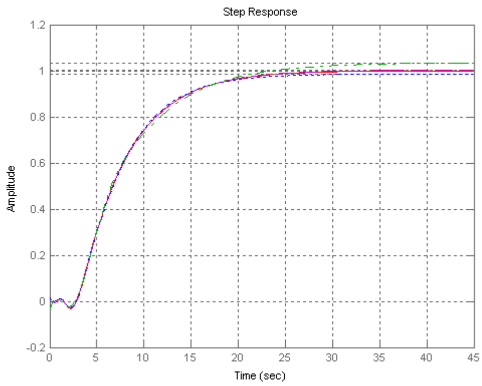

Fig. 7. Unit step response of shift Laguerre-4 approximation with 2nd order weight function: full order (---); reduced 5th order (- -); reduced 4th order (…); reduced 3rd order (-.-.)

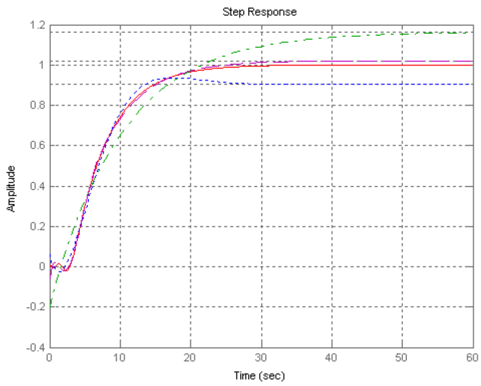

Fig. 8. Unit step response of shift Pade-2 approximation with 2nd order weight function: full order (---); reduced 5th order (- -); reduced 4th order (…); reduced 3rd order (-.-.)

The Hankel norm approximation method gives the system errors in terms of the Hankel norm: the error between the full order and reduced 5th order approximation of $H(s)$ is 0.0026, the error between the reduced 5th order and the reduced 4th order shift Laguerre-4 approximations is 0.0169 and similarly the error between the reduced 4th order and the reduced 3rd order shift Laguerre-4 approximations is 0.0483. It is observed from Fig. 7 that the unit step response difference between the full order and reduced order systems appears in the steady state values and it is very small. Similar results are obtained when the shift Pade-2 approximation is applied to the weighted delay system. The Hankel singular values for the weighted time delay element with the same set of parameter values are obtained as follows: $S = \{0.6823, 0.2584, 0.1156, 0.0583, 0.0223, 0.0036\}$. The unit step time responses of the full order shift Pade-2 approximation and the reduced 5th order, 4th order and 3rd order systems are shown in Fig. 8. It is observed that the error between the system responses increases which is due to the larger values of the Hankel singular values of the shift Pade-2 approximation and therefore the larger Hankel norm difference between the reduced order approximations.

5. Conclusion

The paper considers the problem of the pure time delay element approximation by applying Hankel norm and balanced realizations. The time delay element is modeled in terms of Pade and Kautz series or Laguerre, Kautz and Pade shift operator representations. The error bounds for these different approximation models are discussed and it is shown that time delays can be approximated successfully only when utilizing some weighting functions. The first order and second order weighting functions are used that corresponds to the practice to model industrial processes most often by first order or second order lags and time delay. The Hankel norm and balanced realizations are used to reduce the order of the weighted time delay approximations. The balanced truncation method is a member of the family of Lyapunov type approaches for balanced model reduction. It is based on a numerically efficient algorithm utilizing Cholesky decomposition of both gramians. The Hankel norm approximation uses the standard technique of sequentially reducing the model order by applying balanced realizations and obtaining σ all-pass dilations of the error system. Both methods for model reduction are tested with numerical examples and the corresponding errors are calculated in terms of the Hankel singular values. The results obtained confirm the efficiency of the shift operator approximation of time delay elements in combination with the balanced and Hankel norm model reduction techniques.

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Аппроксимация элемента чистого запаздывания с использованием Ханкеловой нормы и балансовой реализации

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(Р е з ю м е)

В статье рассматривается задача аппроксимации элемента чистого запаздывания с использованием Ханкеловой нормы и балансовой реализации. Взвешанный элемент запаздывания представлен рядами Паде и Каутца или оператором перемещения Лагера, Паде и Каутца. Заданы количественные оценки ошибки аппроксимации чистого запаздывания. Ханкеловые нормы и балансовые реализации исспользуются для определения порядка перерыва аппроксимации. Редуцированная модель сравнивается с аппроксимацией польного порядка для определения эффективности представленных методов.