БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ • BULGARIAN ACADEMY OF SCIENCES

ПРОБЛЕМИ НА ТЕХНИЧЕСКАТА КИБЕРНЕТИКА И РОБОТИКАТА, **56**
PROBLEMS OF ENGINEERING CYBERNETICS AND ROBOTICS, **56**

София • 2006 • Sofia

An Approach for Defining the Displacements of Elastic Link from Open-Loop Kinematic Chain (Manipulator)

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1. Introduction

One of the most important problems in Robotics is the effective real time obtaining of the dynamic equations of motion for the real-time simulation and control purposes, i.e. as well as for effective investigations and manipulator design, so for realizing effective control algorithms. After geometrical description of the concrete structure, the next step is the kinematical modeling, which may be realized on the base of the different parameterizations of the rotation group *SO*(3) and the different algebraic descriptions, using (4×4) homogeneous matrices and (3×3) rotation matrices (see Mladenova [11]). After that, the dynamical modeling of manipulators may be realized on the base on the Lagrange's equations, Newton-Euler recursive equations, the equations of D'Alembert, Gauss, Appel, Kane, etc. We will not referee here the so many papers and books on this subject.

The robot manipulators are divided in some groups: rigid body manipulators (Craig [3]; Angeles [2]; McCarthy [8]; Lilov [5], Lilov and Bojadziev [6], etc.), flexible links manipulators (Schwertassek and Roberson [14]; Soffker [16]; Shabana [15]; Zahariev [18]; Mladenova and Rashkov [10], etc.), manipulators with flexible joints (Ochier, Mladenova and Muller [12]; Mladenova and Muller [9], etc.), and their different combinations. This is according to the structure and the mechanical models. And what about according to computations. In the recent years symbolic computations and parallel algorithms are widely used for efficient modeling and computations. On the one hand the symbolic computations give the possibilities for analytical evaluation, discussions and corrections, on the other hand the parallel algorithms reduce the computational time, which is quite important for on-line simulations and control (see for example, Andreeva and Karastoyanov [1]; Zahariev and Karastoyanov [19], etc.).

Having in mind this study, namely modeling of flexible link manipulators, the main algorithms are based on the following models: the finite element method, the Ritz method and using Lagrange, Newton-Euler and Hamilton equations of motions.

In the present work an approach for defining the displacements of elastic link from open-loop kinematical chain (manipulator) is presented, supposing that the displacements are small. The presented approach is based on the differential equation of an elastic line of a bent beam and the D'Alembert principle. An example of two-link plain manipulator whose second link is flexible is given.

2. Problem statement

Let us consider an example of a cantilever with a fixed point mass at its end (Fig. 1).

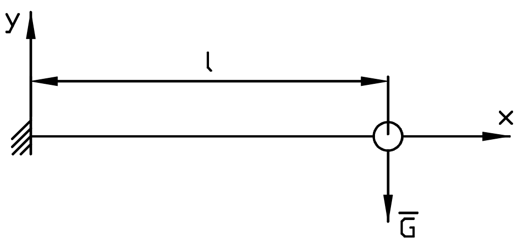

Fig. 1

If the beam stiffness is high enough it could be supposed that the beam is massless, and the only force which loads it is the gravity force of a point mass. We accept the beam stiffness to be big enough if the beam doesn't undergo deformation under the influence only of its own mass. If the displacements are small the elastic line of the beam could be defined as follows (see Timoshenko [17]; Feodosiev [4]):

(1) $$EI_z \frac{d^2 y}{dx^2} = -M_z; \qquad M_z = mg(l - x),$$

where M_z is the bending moment, l is the length of the beam, I_z is the inertia moment of the beam section, m is the mass of the point mass and g is the gravitational constant.

If the beam moves around some fixed point, according to the principle of D'Alembert an additional inertia force $\overline{\Phi}$ could be added (for more details see Loicianski and Lurie [7]). This force is oriented out from the point mass trajectory, it is equal to the product of the mass and its accelerations, and it exists during the whole motion. If the movement is given or known in advance, the inertia force can be found in every moment of time. The displacements of the beam can also be obtained.

The main idea of this algorithm is:

1) to find acceleration of the point mass supposing that the link is undeformable (rigid); this idea is based on the assumption that the elastic displacements are small;

2) to compose an expression of the inertia force of the point mass;

3) to compose a differential equation of the bent elastic line of flexible link;

4) to find the elastic displacements.

The simplest cases of movement of a massless beam with a fixed point mass are analyzed below.

CASE 1. Rotation of a flexible link in vertical plane with constant velocity

Let us consider Fig. 2. A flexible link with fixed point mass rotates around Oz axis in the vertical plane. The mass of the link is neglected and the elastic displacements of the link are accepted to be small. $\overline{\Phi}$ doesn't gene-rate a moment since there is no a vertical component in the movable coordinate system $Ox'y'z'$, i.e.

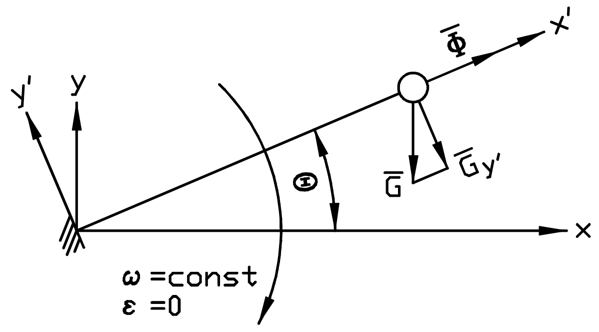

Fig. 2

(2) $$\Phi_{y'} = 0 .$$

The only force which generates a moment with respect to the point of hanging O is the gravity force of the point mass. Its vertical component is

(3) $$G_{y'} = G\cos\theta = mg\cos\theta .$$

The equation of the elastic line is as follows:

(4) $$EI_{z'}\frac{d^2y'}{dx'^2} = -mg\cos\theta(l - x').$$

After integration we have

(5) $$EI_{z'}\frac{dy'}{dx'} = mg\cos\theta\frac{(l-x')^2}{2} + C_1 .$$

The initial conditions are:

(6) $$x' = 0;\ y' = 0;\ \frac{dy'}{dx'} = 0 \Rightarrow C_1 = -\frac{l^2}{2}mg\cos\theta ,$$

or the first integral is

(7) $$EI_{z'}\frac{dy'}{dx'} = mg\cos\theta\frac{(l-x')^2}{2} - \frac{l^2}{2}mg\cos\theta,$$

and after a second integration obtain

(8) $$EI_{z'}y' = -mg\cos\theta\frac{(l-x')^3}{6} - \frac{l^2}{2}mg\cos\theta x' + C_2 .$$

According to the initial conditions the integral constant is

(9) $$C_2 = \frac{l^3}{6}mg\cos\theta .$$

The second integral is

(10) $$y' = -\frac{mg\cos\theta}{EI_{z'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right).$$

So an equation that describes the elastic line of the link at a fixed time moment is obtained. Then during the whole motion the elastic displacements can be described by the following system

$$\theta = \theta(t); \quad y' = -\frac{mg\cos\theta}{EI_{z'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right). \tag{11}$$

Since the angular velocity of the link rotation is constant we have

$$\theta(t) = \alpha t + \beta, \tag{12}$$

where α and β are constants.

The elastic displacements in the absolute coordinate system using the corresponding rotation matrix look like

$${}^{0}\delta = R(z,\theta)\delta', \tag{13}$$

where ${}^{0}\delta = \left[{}^{0}x \;\; {}^{0}y\right]^{\mathrm{T}}$ and ${}^{'}\delta = \left[{}^{'}x \;\; {}^{'}y\right]^{\mathrm{T}}$.

Note. The upper left index shows in which coordinate system we describe the respective quantity.

CASE 2. Rotation of an elastic link in vertical plane with variable velocity

Let us consider Fig. 3. The case is similar to Case 1, but here the movement is accelerating and the angular velocity is not constant. The admissions are the same as in the previous case, but here we have introduced an acceleration constraint, namely ε which is continuous and smooth, i.e. $\dot{\varepsilon}$ exists for every t and it is necessary the first derivative of the acceleration of the mass point to be fixed at every moment. $\overline{\Phi}$ and $\overline{G}$ have vertical components in a movable coordinate system $Ox'y'z'$, hence they generate moments with respect to the hang point of the link. $\Phi_{y'}$ is nothing, but $\Phi_{\tau} = m\varepsilon l$, where $\varepsilon = \varepsilon(t)$ is a known function of the acceleration.

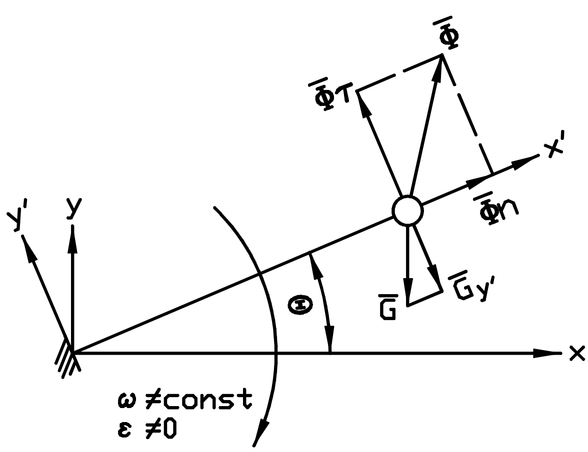

Fig. 3

The equation of the elastic line is

$$EI_{z'}\frac{d^2y'}{dx'^2} = -mg\cos\theta(l - x') + m\varepsilon l(l - x'). \tag{14}$$

This is the same as

$$EI_{z'}\frac{d^2y'}{dx'^2} = (m\varepsilon l - mg\cos\theta)(l - x'), \tag{15}$$

and after integration we obtain

$$EI_{z'}\frac{dy'}{dx'} = -(m\varepsilon l - mg\cos\theta)\frac{(l-x')^2}{2} + C_1 . \tag{16}$$

The initial conditions are the same as before. For C_1 we have

$$C_1 = (m\varepsilon l - mg\cos\theta)\frac{l^2}{2} . \tag{17}$$

The first integral is

$$EI_{z'}\frac{dy'}{dx'} = -(m\varepsilon l - mg\cos\theta)\frac{(l-x')^2}{2} + (m\varepsilon l - mg\cos\theta)\frac{l^2}{2} \cdot \tag{18}$$

Following the same procedure, given in Case 1, it is obtained

$$EI_{z'}y' = (m\varepsilon l - mg\cos\theta)\frac{(l-x')^3}{6} + (m\varepsilon l - mg\cos\theta)\frac{l^2}{2}x' + C_2 . \tag{19}$$

With respect to the initial conditions C_2 is as follows:

$$C_2 = -(m\varepsilon l - mg\cos\theta)\frac{l^3}{6} , \tag{20}$$

and the second integral is

$$y' = \frac{m\varepsilon l - mg\cos\theta}{EI_{z'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right) . \tag{21}$$

The system that describes the elastic displacements of the link throughout the whole motion is

$$\theta = \theta(t); \quad \ddot{\theta} = \ddot{\theta}(t); \quad y' = \frac{m\varepsilon l - mg\cos\theta}{EI_{z'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right) . \tag{22}$$

If the elastic displacements of the link are necessary in absolute coordinate system we may proceed as before using the rotation matrix.

CASE 3. Rotation of the elastic link in horizontal plane with constant velocity

In this case (Fig. 4) the motion is a constant rotation around the immovable axis Oy. Let us attach again a local coordinate system to the link $Ox'y'z'$. The two acting forces are the gravity force $\overline{G}$ and the inertia force $\overline{\Phi}$. The moments are generated by their vertical components:

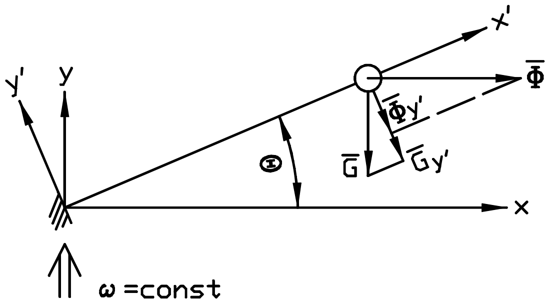

Fig. 4

$$G_{y'} = G\cos\theta = mg\cos\theta;$$

$$\Phi_{y'} = \Phi\sin\theta = ma_O\sin\theta = \\ = m\omega^2 l\cos\theta\sin\theta,$$

where a_O is the centripetal (centrifugal) acceleration of the point mass. In this case θ does not depend on time and appears to be constant during the whole motion.

Of course, $\overline{\Phi}$ and $\overline{G}$ have components along the axis Ox', which also generate moments after beam deformation. They, however, could be neglected provided the components of the forces along axis Ox' are not too high, since the arms of these forces also appear to be small when the displacements are small. However, in spite of the fact there is a rotation that could be immediately measured even by an observer who is located in the local frame $Ox'y'z'$, the situation is the same as if the beam is immovable, and $\overline{\Phi}$ is a kind of a concentrated force having the same direction and magnitude. In this case the elastic beam, after reaching its maximum deformations behaves itself as if it is a rigid body making pure rotation. The equation of the elastic line is

$$EI_{z'}\frac{d^2y'}{dx'^2} = -\Phi_{y'}(l-x') - G_{y'}(l-x'). \tag{23}$$

This equation describes the form of the elastic line of the beam at a fixed time moment. But when $\omega = \text{const}$, then $\Phi = m\omega^2 h = \text{const}$, i.e. the equation is valid for every time moment.

The equation of the elastic line at a fixed time moment is

$$EI_{z'}\frac{d^2y'}{dx'^2} = -mg\cos\theta(l-x') - m\omega^2 l\cos\theta\sin\theta(l-x'). \tag{24}$$

After integration of the last equation it follows:

$$EI_{z'}\frac{dy'}{dx'} = \left(mg\cos\theta + m\omega^2 l\cos\theta\sin\theta\right)\frac{(l-x')^2}{2} + C_1.$$

The initial conditions are the same as before and C_1 is

$$C_1 = -\left(mg\cos\theta + m\omega^2 l\cos\theta\sin\theta\right)\frac{l^2}{2}.$$

The first integral is

$$EI_{z'}\frac{dy'}{dx'} = \left(mg\cos\theta + m\omega^2 l\cos\theta\sin\theta\right)\frac{(l-x')^2}{2} - \left(mg\cos\theta + m\omega^2 l\cos\theta\sin\theta\right)\frac{l^2}{2}, \tag{25}$$

and after a second integration it is obtained

$$EI_{z'}y' = -\left(mg\cos\theta + m\omega^2 l\cos\theta\sin\theta\right)\frac{(l-x')^3}{6} - \left(mg\cos\theta + m\omega^2 l\cos\theta\sin\theta\right)\frac{l^2}{2}x' + C_2.$$

In accordance with the initial conditions the result for the integral constant is

$$C_2 = \left(mg\cos\theta + m\omega^2 l\cos\theta\sin\theta\right)\frac{l^3}{6}.$$

Then the second integral is

$$y' = -\frac{mg\cos\theta + m\omega^2 l\cos\theta\sin\theta}{EI_{z'}}\left(l\frac{x'^2}{2} - \frac{x'^3}{6}\right). \tag{26}$$

Due to the constant velocity of the link rotation, the above equation describes in the local coordinate system the displacements not only at a definite moment, but also throughout the whole motion.

CASE 4. Rotation of the elastic link in a horizontal plane with variable velocity

The case is similar to the previous one, but now the angular velocity of the elastic link is not a constant quantity. The admissions are the same as in the previous case, but here we have introduced an acceleration constrains, namely ε is continuous and smooth, i.e. $\dot{\varepsilon}$ exist for every t and it is necessary the first derivative of the acceleration of the mass point to be fixed at every moment. The movement of the link is rotation around the immovable axis *Oy* with a non-constant velocity, i.e. θ = const and $z \equiv z'$ only at the initial moment. During the motion, the axis *Oy'* describes a circular cone whose axis of symmetry coincides with axis *Oy*. The angle vertex of the cone is 2θ.

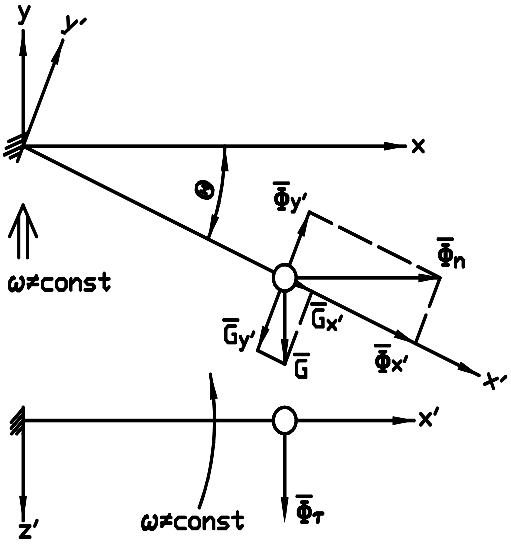

Fig. 5

If the movement is not accelerating, the case will coincide with previous one. Here, however, as a result of the tangential acceleration, we obtain an additional component of the inertia force. This component loads the link of bending in the plane that contains the axis *Ox'* and the acceleration $\bar{a}_\tau$ itself. Let us define the displacements of the point mass in this plane and once we had found the displacements in the vertical plane, containing the gravity force $\bar{G}$ and the centrifugal acceleration $\bar{a}_O$, we could obtain the total displacement.

Thus, in plain *Ox'z'*, the tangential component of the inertia force is

(27) $$\Phi_\tau = ma_\tau = m\varepsilon l\,.$$

In this case the equation of the elastic line of the link in this plane at a fixed time moment is

(28) $$EI_{y'}\frac{d^2z'}{dx'^2} = m\varepsilon l\left(l - x'\right).$$

After integration of the last equation, we obtain

(29) $$EI_{y'}\frac{dz'}{dx'} = -m\varepsilon l\frac{\left(l - x'\right)^2}{2} + C_1\,.$$

The initial conditions are

(30) $$x' = 0;\;\; z' = 0;\;\; \frac{dz'}{dx'} = 0\,.$$

For C_1 it is obtained

$$C_1 = m\varepsilon \frac{l^3}{2}.$$

The first integral is

(31) $$EI_{y'} \frac{dz'}{dx'} = -m\varepsilon l \frac{(l-x')^2}{2} + m\varepsilon \frac{l^3}{2}.$$

After a second integration, namely

(32) $$EI_{y'} z' = m\varepsilon l \frac{(l-x')^3}{6} + m\varepsilon \frac{l^3}{2} x' + C_2 ,$$

and in accordance with the initial conditions, the corresponding integral constant is

(33) $$C_2 = -m\varepsilon \frac{l^4}{6},$$

or the second integral is

(34) $$z' = \frac{m\varepsilon l}{EI_{y'}} \left(l \frac{x'^2}{2} - \frac{x'^3}{6} \right).$$

The form of the elastic line in plain *Ox'y'* we could obtain as in the previous case. The difference here is that the quantity ω is not constant and the direction of $\overline{\Phi}_{y'}$. So, similar to CASE 3 we have

(35) $$y' = -\frac{mg\cos\theta - m\omega^2 l\cos\theta\sin\theta}{EI_{z'}} \left(l \frac{x'^2}{2} - \frac{x'^3}{6} \right).$$

The form of the elastic line during the whole motion is described by the following system:

(36) $$\begin{aligned} &\theta = \theta(t); \quad \dot{\theta} = \dot{\theta}(t); \quad \ddot{\theta} = \ddot{\theta}(t) \\ &z' = \frac{m\varepsilon l}{EI_{y'}} \left(l \frac{x'^2}{2} - \frac{x'^3}{6} \right) \\ &y' = -\frac{mg\cos\theta - m\omega^2 l\cos\theta\sin\theta}{EI_{z'}} \left(l \frac{x'^2}{2} - \frac{x'^3}{6} \right). \end{aligned}$$

The total displacement is

(37) $$'\delta = \sqrt{'\delta_{z'}^2 + '\delta_{y'}^2},$$

and the direction cosines are respectively:

(38) $$\gamma = \frac{'\delta_{z'}}{\sqrt{'\delta_{z'}^2 + '\delta_{y'}^2}}; \qquad \beta = \frac{'\delta_{y'}}{\sqrt{'\delta_{z'}^2 + '\delta_{y'}^2}}.$$

3. General motion of a manipulator

In this section we consider a general motion of a massless elastic link with a fixed point mass in its end which link appears to be the last one in a structure of open-loop kinematic chain (manipulator). We have in mind an open-loop kinematic chain (manipulator) that consists of n joints with one degree of freedom and $n + 1$ links (the base is denoted by 0). The last link (link n) is flexible, the rest ones are rigid. A point mass is immovably connected to the free end of the last link. A coordinate system oriented according to the Denavit-Hartenberg notation, is attached to each link. It is convenient the axes of a coordinate system numbered to be oriented along the principal inertia axes of the link. In case the above mentioned system doesn't coincide with the respective one of link n, oriented according to notation of Denavit-Hartenberg, the transformation matrix between the two coordinate systems is to be composed. Let us suppose further that both systems coincide. In this right-hand frame ($Ox'y'z'$) coinciding with coordinate system n, we calculate the elastic displacements. The movement of the link could be calculated in the inertial coordinate system $Oxyz$, which is attached to the base of a manipulator. Axis Ox' appears to be a tangent to the elastic line of the link in the hang point of the link after its deformation; axis Oz' is perpendicular to axis Ox' and is oriented along the rotation (translation) joint axis. With respect to the local coordinate system of the elastic link, its joint end is immovable and doesn't complete elastic displacements, since it is connected to the previous link, which is rigid. On the other hand, the actual displacements of the point mass are due to its absolute acceleration and its weight. This procedure is described further down in 10 steps (elements of this procedure could be found also in Craig [3], Rashkov and Mladenova [13], Mladenova and Rashkov [10]).

1. What we enter first are the geometrical and mechanical characteristics of the links: size, mass of point mass, elasticity module of link n.

2. The next quantities that are entered are the laws of motion of the links (joint variables) together with cycle of the manipulator (the duration of the motion of the manipulator).

3. Further the transformation matrices are composed. The general transformation matrix is

$$
{}^{i-1}_{\;\;i}T = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i\cos\alpha_{i-1} & \cos\theta_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1}d_i \\ \sin\theta_i\sin\alpha_{i-1} & \cos\theta_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{39}
$$

where θ, a, α, d are the Denavit-Hartenberg joint variable and link parameters (joint angle, link length, link twist and link offset).

4. The angular and linear velocities are calculated:

(a) for joint i +1 rotational, they are

$$
\begin{gathered}
{}^{i+1}\omega_{i+1} = {}^{i+1}_{\;\;i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}Z_{i+1}, \\
{}^{i+1}v_{i+1} = {}^{i+1}_{\;\;i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right);
\end{gathered} \tag{40}
$$

(b) for joint i +1 prismatic they are

(41) $$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i\,, $$
$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}Z_{i+1}\ , $$

where ${}^{i+1}_{i}R$ is an inverse rotation matrix with dimension (3×3), ${}^{i}P_{i+1}$ – the distance from the origin of coordinate system $\{i\}$ to the origin of coordinate system $\{i + 1\}$, ${}^{i+1}Z_{i+1}$ – unit vector along the axis Z, i.e. $\dot{\theta}_{i+1}\,{}^{i+1}Z_{i+1} = {}^{i+1}\begin{bmatrix}0 & 0 & \dot{\theta}_{i+1}\end{bmatrix}^{\mathrm{T}}$.

5. The angular and linear accelerations are calculated:

(a) for joint i + 1 rotational, they are

(42) $$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}Z_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}Z_{i+1}, $$
$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right]; $$

(b) for joint i + 1 prismatic, they are

(43) $$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i\,, $$
$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right] + 2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}Z_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}Z_{i+1}\ . $$

6. The inertia force of the point mass is composed:

(44) $$ {}^{n}\Phi_n = m\,{}^{i+1}\dot{v}_{i+1} $$

7. The gravity force of the point mass with respect to coordinate system n is calculated:

(45) $$ {}^{n}G_n = \left(\prod_{i=1}^{n} {}^{\;\;i}_{i+1}R\right)^{T} {}^{0}G_n\ . $$

8. The differential equations of the elastic line in the two plains are given:

(46) $$ EI_{z'}\frac{d^2y'}{dx'^2} = \sum M_{z'} = \left({}'\Phi_{y'} + {}'G_{y'}\right)\left(l - x'\right), $$
$$ EI_{y'}\frac{d^2z'}{dx'^2} = \sum M_{y'} = \left({}'\Phi_{z'} + {}'G_{z'}\right)\left(l - x'\right). $$

9. The displacements in the two plains are found and the total displacement is defined:

(47) $$ {}'\delta = \sqrt{{}'\delta_{z'}^2 + {}'\delta_{y'}^2}\ , $$

and the direction cosines are respectively

(48) $$ \gamma = \frac{{}'\delta_{z'}}{\sqrt{{}'\delta_{z'}^2 + {}'\delta_{y'}^2}}, \qquad \beta = \frac{{}'\delta_{y'}}{\sqrt{{}'\delta_{z'}^2 + {}'\delta_{y'}^2}}\ . $$

10. If it is necessary to find the displacements with respect to absolute coordinate system we could write

(49) $$ {}^{0}P_n = \prod_{i=1}^{n} {}^{i-1}_{\;\;i}T\;{}^{n}P_n\,, $$

where ${}^{n}P_{n}$ is a 4×1 vector and represents the elastic displacement of a characteristic point (for example an end-effector) from the last link n in time and ${}^{i-1}_{i}T$ is a 4×4 homogeneous transformation matrix that represents the description of frame $\{i\}$ relative to $\{i-1\}$.

4. Example

The manipulating system (Fig. 6) consisting of two movable links and two rotational joints with one degree of freedom is presented. To the second link at its free end a point mass is immo-vably attached. The characteristics of the manipulators are as follows:

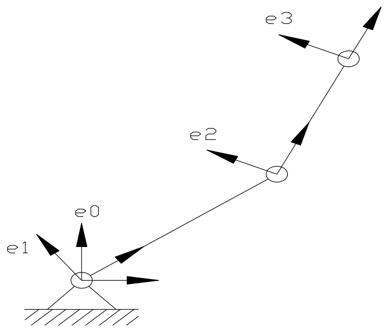

Fig. 6. 2R planar manipulator

- length of link 1: $L[1]$ = 1 m;
- length of link 2: $L[2]$ = 0.5 m;
- mass of the point mass at the end of the link 2: m = 5 kg;
- the section of link 2 is a circle with radius: r = 0.01 m;
- elasticity module of link 2 (steel): $E = 2.1\times10^{11}$ Pa.

As exemplary laws of motion of the joint angles we have chosen 5th degree polynomials:

$$\theta[1](t) = -\frac{5t^2}{2} + \frac{(1840+20\pi)t^3}{2000} + \frac{(-2220-30\pi)t^4}{20\,000} + \frac{(880+12\pi)t^5}{200\,000},$$
$$\theta[2](t) = -\frac{\pi}{2} - \frac{5t^2}{2} + \frac{(1840+20\pi)t^3}{2000} + \frac{(-2220-30\pi)t^4}{20\,000} + \frac{(880+12\pi)t^5}{200\,000}$$

with duration of time 10 s.

The biggest displacements are obtained at the end of the elastic link where the point mass is attached, i.e. the point with coordinates (0.5, 0, 0) with respect to $Ox'y'z'$.

Fig. 7 shows the mode of the vertical components of the inertia force (continuous line) and of the gravity force of the point mass during the motion, i.e. the components

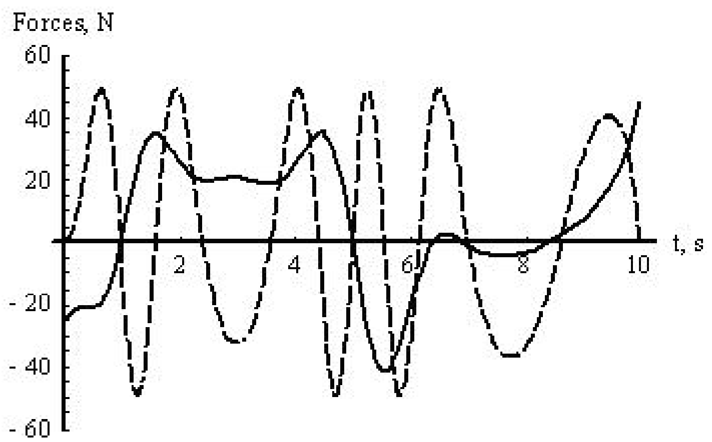

Fig. 7. $G_{y'}$ (- - -) and $\Phi_{y'}$ (—) depending on time t

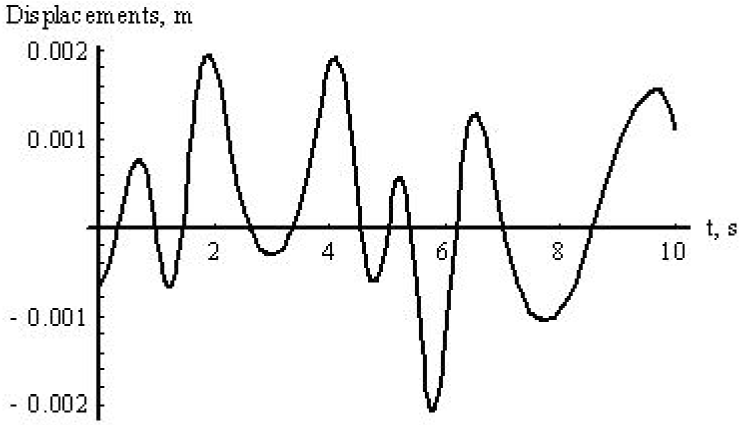

Fig. 8. The displacements of the elastic link (where the point mass is fixed) during the motion

$G_{y'}$ and $\Phi_{y'}$ depend on time t. Fig. 8 shows the displacements of the elastic link (where the point mass is fixed) during the motion. The maximum positive and negative displacements of the point mass are respectively 0.0019 m when t = 1.89 s and 0.002 when t = 5.8 s with the link length 0.5 m. These results coincide with the assumption that the displacements are small.

5. Conclusion

The considered algorithm can be used also in cases of more than one point masses attached to the flexible link arbitrary, as well as in cases of distributed loads, concentrated forces and moments. The mass of the link can also be included. It is also possible a contact points of the forces and concentrated moments, as well as the length of the link, to be changed with respect to time. In its base part the algorithm doesn't change. It is only necessary to correct the differential equation of the elastic line of link n.

In this algorithm a check is also possible of the maximum normal stresses of the elastic link to be included, for example the quotient of the maximum bending moment and the moment of resistance to be calculated. However, it is necessary to take account of the force cyclicity since only this kind of check could not be a strength criterion.

To minimize the vibrations of the elastic link there shouldn't be jerk change of the acceleration, i.e. the acceleration to be fixed at each time moment, except in trivial cases, when the acceleration is constant or doesn't exist.

The presented algorithm is valid only for cases of small displacements. When the displacements are large, a contact point of the forces move during the link deformation which have to be taken into account. Besides, equation (1) is obtained on the assumption that the quantity dy/dx is small compared to unity. In order to obtain large displacements it is necessary changes of the link curve to be considerable. But if the strains are not bigger than the limit of elasticity that is possible only with small height of the section, i.e. the link should be in the form of a thin band or a thin wire.

The algorithms for generating laws of motion, kinematics of the manipulator and for calculating of the elastic displacements are modelled as software packages (Mathematica Package) in Mathematica 4.2. More information can be found in (R a s h k o v and M l a d e n o v a [13]).

Acknowledgements. The second author would like to thank the Alexander von Humboldt Foundation for donating the computer algebra system Mathematica 4.2 which was indispensable in realizing the above programme algorithm in symbolic form. This work is also partially supported by NSF, Grant No. TH-1411/2004.

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Прием исследования упругого звена в составе открытой кинематической цепи (манипулятора)

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(Р е з ю м е)

Рассматривается приём исследования упругого звена в составе открытой кинематической цепи (манипулятора) в случае небольших пере-мещений. Этот приём базирован на дифференциальном уравнении изогнутой оси упругой балки и на принципе Даламбера. Закон движения рассматривается совместно с уравнением изогнутой оси. Анализируются случаи самого простого движения упругого звена, имеющего тяжёлую точку, неподвижно прикреплённую к его концу. Кроме этого, рассматривается общий случай движения того же самого звена, которое звено является последним в составе любого манипулятора. Приведен пример равнинного манипулятора, состоящегося из двух звен, второе из которых является упругим.