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Kinematics and Force Analysis of a Five-Link Mechanism by the Four Spaces Jacoby Matrix

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1. Introduction

The five-link planar manipulative system (MS), shown in Fig. 1, contains only rotational joints. Some parts of them are passive, the remaining – active. All the bodies could change their dimensions in some borders [4] during the design process and in that way the features of the MS change. The body *1* (l_1) is more particular as it stays immobile (it represents the support). The bodies *2* and *5* are driving bodies. With the help of appropriate rotation of the actuating bodies, the characteristic point *B* of the MS can follow desired planar trajectory in the borders of the working zone.

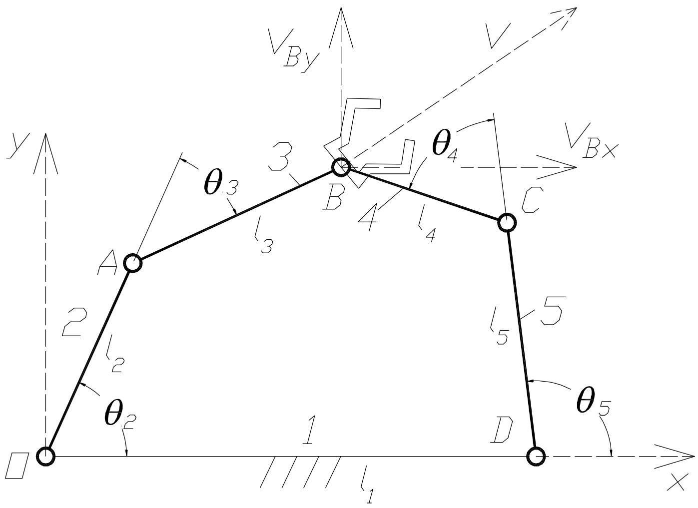


Fig. 1. Structural scheme of the considered manipulative system

The velocity $V = [V_{B_x}, V_{B_y}]^T$ of the characteristic point B is determined through the angular velocities $\dot{\theta} = [\dot{\theta}_2, \dot{\theta}_5]^T$ of the bodies *2* and *5* and depends on the transfer function of the mechanism. Usually the transfer function is described by the Jacoby matrix J:

(1) $$V = J\dot{\theta}.$$

This expression is known as forward kinematics problem and for the considered MS could be solved using different approaches [3]. The analytical symbolic solution could be particularly useful for making several conclusions concerning the singular configurations of the MS as well as the MS metric. The classical approach [2] for solving such kind of problems requires the solution of the standard position task (forward kinematics) $f(\theta_i) = X$, $i = 2, 5$; $X = V = [B_x, B_y]^T$ or of the inverse kinematics. After that the obtained results are differentiated with respect to the general coordinates $\theta = [\theta_2, \theta_5]^T$. In that concrete example such a solution is complex and ambiguous (in the general case). The forward kinematics (standard position task) has two solutions, the inverse – four. These arguments determine the necessity to search for other approaches for the analytical solution of the forward kinematics (position task).

2. The Jacoby matrix for the closed loop manipulative system

Let's assume that the MS is divided into two parts representing two open planar kinematics chains with two links (Fig. 2).

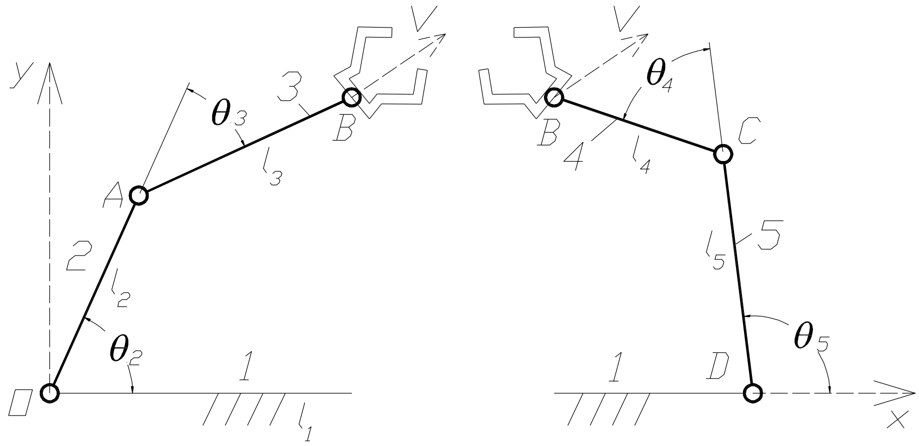


Fig. 2. Representation of the MS from Fig. 1 as a system containing two open structures

The matrix of Jacoby $J_{1,2}$ for each of them is known [3]. For the left (J_1) MS we obtain

(2) $$J_1 = \begin{bmatrix} -A_{11} & -A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where:

$$A_{11} = l_2 \sin\theta_2 + l_3 \sin(\theta_2 + \theta_3)\,,\ A_{12} = l_3 \sin(\theta_2 + \theta_3)\,,$$
$$A_{21} = l_2 \cos\theta_2 + l_3 \cos(\theta_2 + \theta_3)\,,\ A_{22} = l_3 \cos(\theta_2 + \theta_3)\,.$$

Analogously we can obtain for the right system:

(3)
$$J_2 = \begin{bmatrix} -B_{11} & -B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where:

$$B_{11} = l_5 \sin\theta_5 + l_4 \sin(\theta_5 + \theta_4)\,,\ B_{12} = l_4 \sin(\theta_5 + \theta_4)\,,$$
$$B_{21} = l_5 \cos\theta_5 + l_4 \cos(\theta_5 + \theta_4)\,,\ B_{22} = l_4 \cos(\theta_5 + \theta_4)\,.$$

If we admit that the distance between both systems is l_1, and that they reach one and the same point B, and also the velocity V of that point B reached by the first and the second MS is the same, we obtain the system

(4)
$$\left| \begin{aligned} V_{B_x} &= -A_{11}\dot{\theta}_2 - A_{12}\dot{\theta}_3 \\ V_{B_y} &= A_{21}\dot{\theta}_2 + A_{22}\dot{\theta}_3 \\ V_{B_x} &= -B_{11}\dot{\theta}_5 - B_{12}\dot{\theta}_4 \\ V_{B_y} &= B_{21}\dot{\theta}_5 + B_{22}\dot{\theta}_4 \end{aligned} \right.$$

or in a matrix form:

$$\begin{bmatrix} V \\ V \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} . \begin{bmatrix} \dot{\theta}_{2,3} \\ \dot{\theta}_{5,4} \end{bmatrix},$$

where $\dot{\theta}_{2,3} = [\,\dot{\theta}_2, \dot{\theta}_3]^T$ and $\dot{\theta}_{5,4} = [\,\dot{\theta}_5, \dot{\theta}_4]^T$.

Eliminating the angular velocities $\dot{\theta}_3$ and $\dot{\theta}_4$ in the passive joints for the forward kinematics problem we obtain

(5)
$$\left| \begin{aligned} V_{B_x} &= -C_{11}\dot{\theta}_2 - C_{12}\dot{\theta}_5 \\ V_{B_y} &= C_{21}\dot{\theta}_2 + C_{22}\dot{\theta}_5 \end{aligned} \right.$$

or in a matrix form

$$V_C = J\dot{\theta}\,,\quad J = \begin{bmatrix} -C_{11} & -C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where:

$$C_{11} = A_{11} + A_{12}\frac{(B_{12}A_{21} - A_{11}B_{22})}{(A_{12}B_{22} - B_{12}A_{22})}\,,\ C_{12} = A_{12}\frac{(B_{11}B_{22} - B_{21}B_{12})}{(A_{12}B_{22} - B_{12}A_{22})}\,,$$

$$C_{21} = A_{21} + A_{22}\frac{(B_{12}A_{21} - A_{11}B_{22})}{(A_{12}B_{22} - B_{12}A_{22})}, \quad C_{22} = A_{22}\frac{(B_{11}B_{22} - B_{21}B_{12})}{(A_{12}B_{22} - B_{12}A_{22})}.$$

The coefficients $C_{i,j}$ could be determined if $A_{12}B_{22} - B_{21}A_{22} \neq 0$.

3. Spaces of the Jacoby matrix

It is known [1, 2], that every matrix defines four spaces which dimensions are determined by the matrix rank and order. Further we will consider the physical and geometric interpretation of these spaces related with the Jacoby matrix for manipulative systems.

3.1. Column space (image) of the Jacoby matrix $\Re(J)$

It transforms (1), the area of admissible values of the controlled velocity vectors of the actuating bodies $\dot{\theta}$ = [$\dot{\theta}_2$, $\dot{\theta}_5$]$^{\mathrm{T}}$, in corresponding velocities of the end-effector $V = [V_{B_x}, V_{B_y}]^{\mathrm{T}}$. This space dimension is equal to r – the rank of the Jacoby matrix (or the number of the matrix independent columns). In this concrete case, the two-dimensional space of the angular velocities generates a two-dimensional space of the velocity of the end-effector. There exist some robot states or configurations (particular or singular) for which the coefficients C_{ij} corresponding to J are such that the two-dimensional space of the generalized velocities generates one-dimensional space of the absolute velocity of the point B (r =1).

3.2. Row space $\Re(J^{\mathrm{T}})$

The row space of the matrix J coincides with the column space of J^{T}. With the help of this space we can determine the actuating moments $\tau = [M_2, M_5]^{\mathrm{T}}$, which must be created in the actuating joints *2* and *5* to equilibrate the external forces $F = [F_x, F_y]^{\mathrm{T}}$, applied to the end-effector:

(6) $$\tau = J^{\mathrm{T}}F.$$

The friction forces and other losses are not considered.

3.3. Zero space of J (Ker(J))

It is defined by the system $Jx = 0$ and its dimension is $n - r$, where n is the number of rows of the matrix J. It describes this subspace of vectors of angular velocities $x \in \dot{\theta}_i$, which does not generate velocities V in the end-effector.

3.4. Zero space of J^{T}(Ker(J^{T})) – left zeros of J

It is defined by the system $J^{\mathrm{T}}y = 0$ and its dimension is $m - r$, where m is the number of columns of J. It describes this set of vectors of external forces $y \in F_i$, for which there is no need of motors equilibrating torques $\tau = [M_2, M_5]^{\mathrm{T}}$,.

The zero space is also known as a kernel of the matrix. It is obvious that the zero vectors $x = [0, 0]^T$ belong to J and J^T. It is known that the defect γ of the matrix denotes the difference between the higher value of the rows or columns number of the matrix J and its rank [1, 2]: $\gamma = \max(m, n) - r$. In our case the maximal possible defect of J is $\gamma = 2$ and it is obtained when the rank of the matrix is zero $r = 0$.

4. Singular configurations

It is very important to define the rank r of J for the analysis of MS [2]. It is equal to the number of independent rows (columns) of the matrix and can be determined by calculating the matrix determinant (if it exists) and its minors (sub-matrices determinants). We are searching for configurations where $\det(J) = 0$ (the rank of J decreases). These configurations are known as singular and the MS changes its features in such configurations. The four spaces of the Jacoby matrix change their dimensions.

Statement 1. The determinant of the matrix J (for the MS in Fig. 1) is equal to zero only if the determinant of $J_1(2)$ or the determinant of $J_2(3)$ is zero:

$$\det(J) = 0 \Leftrightarrow \left| \begin{array}{l} \det(J_1) = 0, \\ \det(J_2) = 0, \end{array} \right. \tag{7}$$

J_1 and J_2 are the corresponding Jacoby matrices for the left and right open chains (Fig. 2) of the five-link closed MS (Fig. 1).

Statement 2. The determinant of the matrix J (for the MS – Fig. 1) tends to infinity when

$$\theta_2 + \theta_3 = \theta_4 + \theta_5 . \tag{8}$$

The demonstration of both statements 1 and 2 is accomplished as follows:

We obtain the determinant of J as:

$$\det(J) = -C_{11}C_{22} + C_{21}C_{12} . \tag{9}$$

After some transformations it can be written:

$$\det(J) = \frac{(A_{11}A_{22} - A_{21}A_{12})(B_{11}B_{22} - B_{21}B_{12})}{(B_{12}A_{22} - A_{12}B_{22})}, \tag{10}$$

or

$$\det(J) = \frac{\det(J_1)\det(J_2)}{l_4 \sin(\theta_5 + \theta_4) l_3 \cos(\theta_2 + \theta_3) - l_4 \cos(\theta_5 + \theta_4) l_3 \sin(\theta_2 + \theta_3)} .$$

From the last we obtain

$$\det(J) = \frac{\det(J_1)\det(J_2)}{l_3 l_4 \sin(\theta_5 + \theta_4 - \theta_2 - \theta_3)} . \tag{11}$$

It is obvious the determinant becomes equal to zero when some of the multipliers in the nominator of (11) take zero values. When the denominator tends to zero then the determinant of J tends to infinity. If l_3 and l_4 lengths are different from zero, the last comes true only if:

(12) $$\theta_5 + \theta_4 - \theta_2 - \theta_3 = 0 \pm k\pi \text{ , } k = 1, 2, \ldots$$

Corollary 1. When the force transformation angle $\psi \leq$ ABC(Fig.1) [4] between the bodies *3* and *4* tends to zero (or 180°), then the determinant of J tends to infinity. In that case we need extremely great actuating torques to equilibrate the external forces acting on the end-effector. The demonstration of the corollary 1 could be easily done, taking into account that the sum of the internal angles of the tetragon is equal to 360° (2π rad). It is obvious that when $\psi = 0$, the mechanism forms a tetragon. For the sum of its internal angles we obtain: $\theta_2 + \theta_3 - \pi + \pi - \theta_4 + \pi - \theta_5 = 2\pi$, and therefore $\theta_2 + \theta_3 - \theta_4 - \theta_5 = \pi$. Condition (12) is satisfied.

5. Numeric examples

5.1. Example 1

General case: A manipulative system is considered which bodies lengths are (Fig. 3) l_1=0.1, l_2=0.2, l_3=0.25, l_4=0.35, l_5=0.1 (Fig. 3); θ_2=100.03°; θ_3= –52.9°; θ_4=79.08°; θ_5=20.53°. Then we obtain:

$$J_1 = \begin{bmatrix} -0.38 & -0.183 \\ 0.135 & 0.17 \end{bmatrix}, \ \det(J_1) = 0.04; \quad J_2 = \begin{bmatrix} -0.38 & -0.345 \\ 0.035 & -0.058 \end{bmatrix}, \ \det(J_2) = 0.034;$$

$$J = \begin{bmatrix} -0.198 & -0.091 \\ -0.034 & 0.084 \end{bmatrix}, \ \det(J) = -0.02.$$

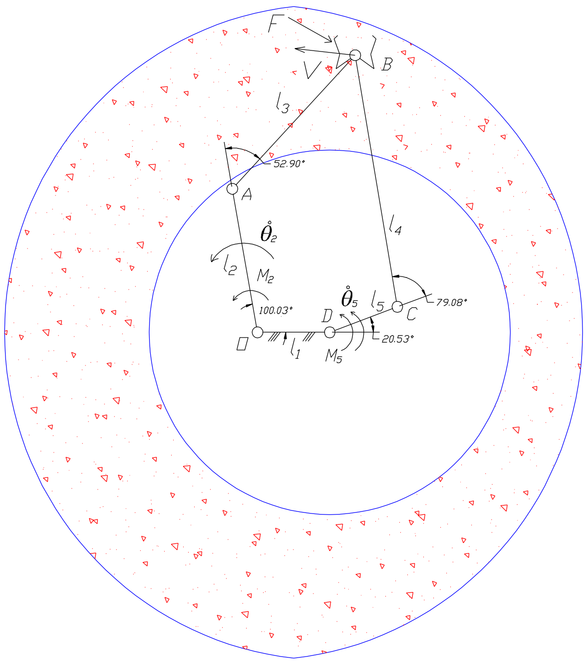


Fig. 3. MS in arbitrary configuration – example 1

Working configuration of MS. It is possible to realize some motion (and also forces) in the plane in arbitrary direction. The coefficients of J are transfer values for the concrete configuration of the mechanism.

5.2. Singular case with defect $\gamma = 1$ for the matrix J – example 2

The bodies lengths are the same as in the example 1 and the generalized coordinates are (Fig. 4): θ_2=57.38°; θ_3=0°; θ_4=63.27°; θ_5=18.86°. Then we obtain:

$$J_1 = \begin{bmatrix} -0.379 & -0.211 \\ 0.243 & 0.135 \end{bmatrix}, \ \det(J_1) = 0\,; \ J_2 = \begin{bmatrix} -0.379 & -0.347 \\ 0.143 & 0.048 \end{bmatrix}, \ \det(J_2) = 0.031,$$

$$\det(J) = 0\,.$$

In this configuration if $\dot{\theta}_5 = 0$ the realization of any velocities does not generate the end-effector velocity, i.e. the vectors $\dot{\theta} = [\dot{\theta}_2, 0]^{\mathrm{T}}$ belongs to the zero space (Ker(J)) of J. Forces acting in the direction $F = k\begin{bmatrix} 1 & \dfrac{0.18}{0.115} \end{bmatrix}^{\mathrm{T}}$ (where k is a real number) cannot be equilibrated by the actuating torques. They belong to the zero space (Ker(J^{T})) of J^{T} and are absorbed by the links of the MS. The maximal force in that direction that can be supported by the construction depends on the robustness of the elements. Such kind of singularities could be observed in mechanisms with any metrics (arbitrary proportion between body lengths, for which the mechanism is defined [4]). At least one of the two open chain MS has configurations where its determinant J_1 (or J_2) becomes zero. The forward kinematics problem has two solutions and at least one of them is singular. The points from the working zone, where $\gamma = 1$, are on its borders. The inverse kinematics problem for them has two singular solutions.

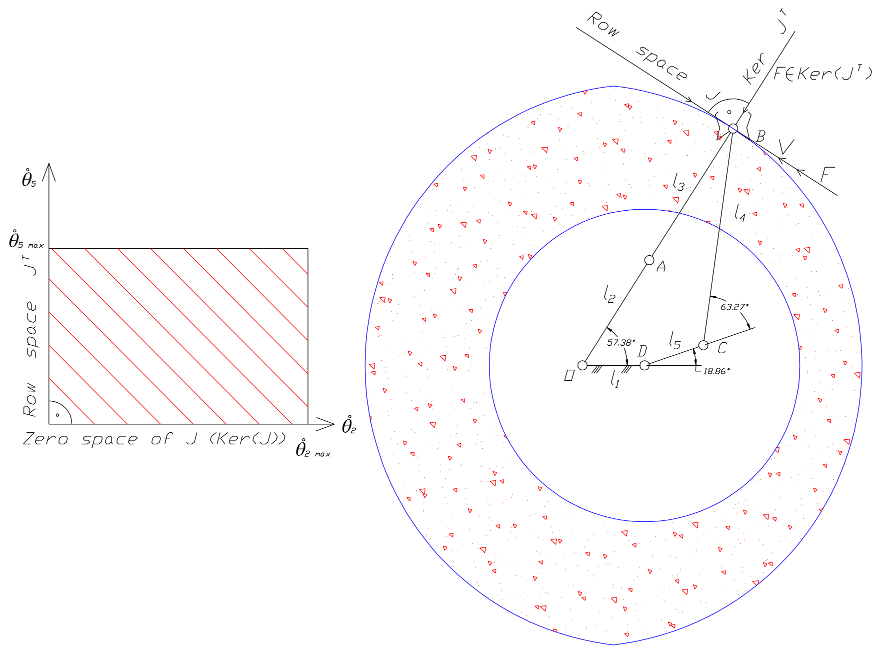


Fig. 4. Singular case with defect $\gamma = 1$ for the matrix J – example 2

5.3. Singular case with defect $\gamma = 2$ for the matrix J – example 3

The links lengths are the same as in the example 1 and the generalized coordinates are (Fig. 5): θ_2=83.62°; θ_3=0°; θ_4=0°; θ_5=96.38°. Then we obtain:

$$J_1 = \begin{bmatrix} -0.447 & -0.248 \\ 0.05 & 0.028 \end{bmatrix}, \ \det(J_1) = 0; \quad J_2 = \begin{bmatrix} -0.447 & -0.348 \\ 0.05 & 0.039 \end{bmatrix}, \ \det(J_2) = 0,$$

$$J = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \det(J) = 0.$$

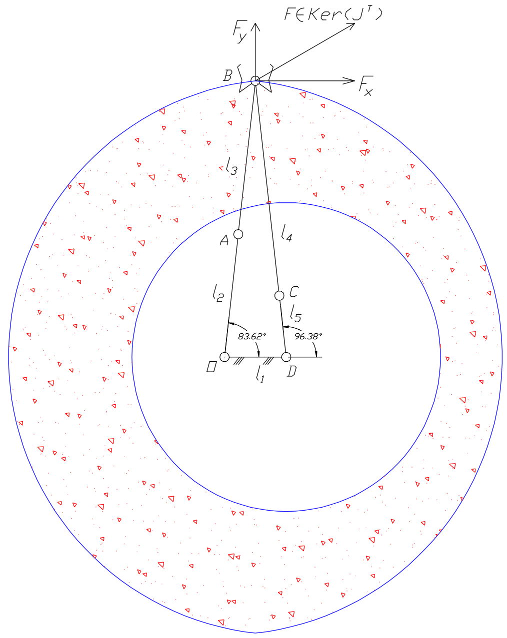


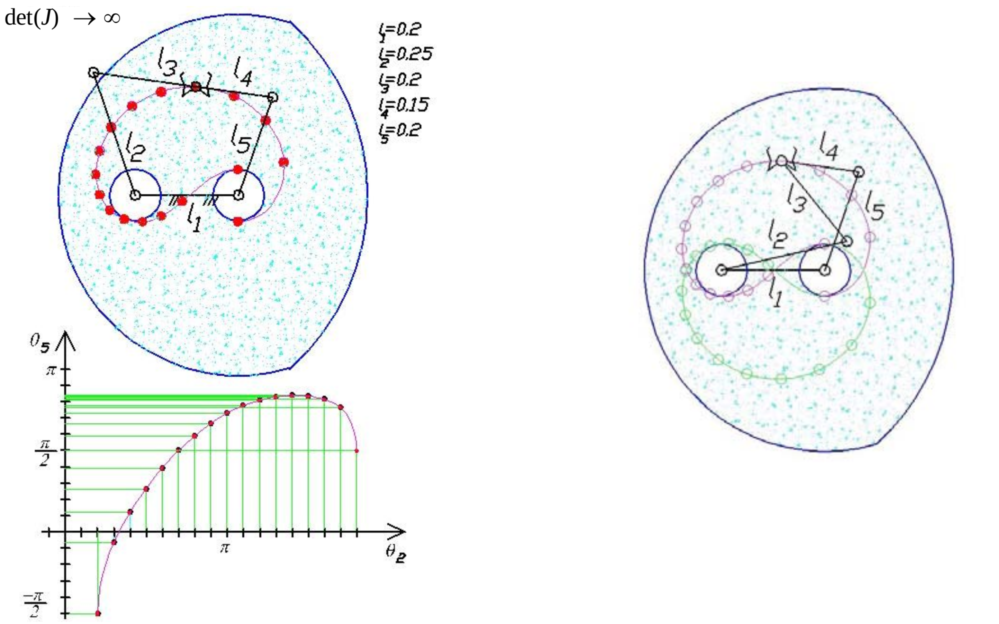
Fig. 5. Singular case with defect $\gamma = 2$ for the matrix J – example 3

The forward and inverse kinematics problems have a unique solution. If $l_1 \neq 0$ there exist only one or two points, where $\gamma = 2$. In such a configuration the MS is extremely stable with respect to the forces applied on the end-effector. Their equilibration is realized only by the links and the supports and is not transferred to the actuating devices (Ker(J^T)$\in F = [F_x, F_y]^T$). It becomes difficult to control the velocity of the point B. The zero space Ker(J) coincides with all the plane $\dot{\theta} = [\dot{\theta}_2, \dot{\theta}_5]^T$.

5.4. Singular case, where det(J) tends to infinity – example 4

A manipulative system which links length are l_1=0.2, l_2=0.25, l_3=0.2, l_4=0.15, l_5=0.2 is considered (Fig. 6); θ_2=114.71°; θ_3= –119.96°; θ_4=97.47°; θ_5=77.28°.
Then it can be written:

$$J_1 = \begin{bmatrix} -0.209 & -0.018 \\ 0.095 & 0.199 \end{bmatrix}, \ \det(J_1) = -0.043; \quad J_2 = \begin{bmatrix} -0.209 & -0.014 \\ -0.105 & -0.149 \end{bmatrix}, \ \det(J_2) = 0.03,$$



a) Five-link mechanism in singular configuration where $\det(J) \to \infty$

b) Reaching the same point without falling in a singular configuration

Fig. 6. Singular case, where det(J) tends to infinity – example 4

$$J = \begin{bmatrix} \infty & -\infty \\ \infty & -\infty \end{bmatrix}, \ \det(J) \to \infty .$$

In Fig. 6a the corresponding graphical solution in generalized and Cartesian coordinates is presented. In this configuration a part of the forces acting on the end-effector (point B) cannot be equilibrated with the help of the actuating torques. Such types of singular configurations are placed inside the working zone of the MS. During the control of the MS such configurations must be avoided due to decreasing of functional capabilities. The end-effector can reach these points (denoted by o), passing by both – singular or nonsingular – configurations (for instance Fig. 6b). The forward kinematics problem for such a point has 4 solutions, but only for one of them (Fig. 6a)) $\det(J) \to \infty$.

5.5. Singular case – indefiniteness of type 0/0 – example 5

The following MS will be considered, which links lengths are (Fig. 7): l_1=0.25, l_2=0.3, l_3=0.2, l_4=0.15, l_5=0.1; θ_2=109.47°; θ_3=-148.41°; θ_4=0°; θ_5=141.06°. Then it could be written:

$$J_1 = \begin{bmatrix} -0.157 & 0.126 \\ 0.056 & 0.156 \end{bmatrix}, \ \det(J_1) = -0.031; \quad J_2 = \begin{bmatrix} -0.157 & -0.094 \\ -0.194 & -0.117 \end{bmatrix}, \ \det(J_2) = 0,$$

$$J = \begin{bmatrix} \infty & \frac{0}{0} \\ \infty & \frac{0}{0} \end{bmatrix}, \ \det(J) = \frac{0}{0}.$$

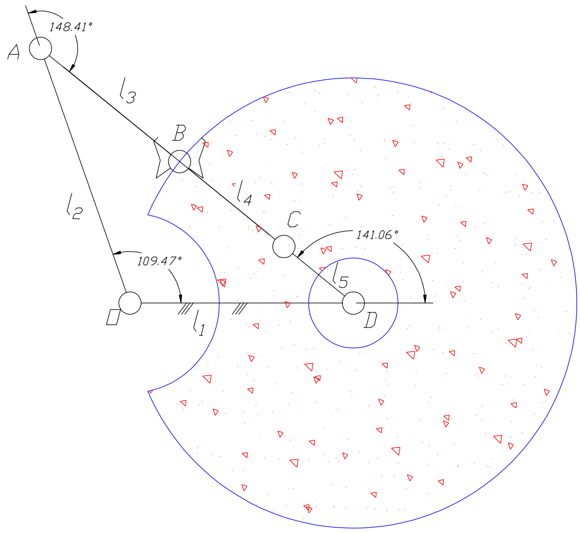


Fig. 7. Singular case – indefiniteness of type 0/0 – example 5

In similar situations the possibility for realization of motions and forces as well as the control of the MS is extremely difficult.

6. Conclusion

The represented approach for the Jacoby matrix determination leads to the demonstration of statements 1 and 2. Consequently the singular configurations of the MS can be easily detected and some conclusions concerning the mechanism behavior in these configurations can be formulated. The corresponding singular configurations are realizable for different proportions of the mechanism link lengths [4, 5]. The case when the determinant of J tends to infinity is particular and in that sense the configurations for which this condition is satisfied could be considered as singular. The interpretation of the Jacoby matrix spaces is physically useful as well in the process of synthesis and design of the manipulative system as for the control process. The extreme values of the transfer function (the elements of J) are used to determine the maximal loads of the actuating mechanisms of the MS.

The main disadvantages of the proposed method are:

- We cannot determine the reactions in the passive joints as well as the reactions generated by forces and torques, which cannot be stabilized by the actuating devices;
- it does not take into account the friction losses.

The advantages are:

- easy determination of the Jacoby matrix and its determinant;
- simple physical interpretation of the singular configurations as a result of their reduction as a combination of two already known and well studied MS;
- the symbolic writing of the transfer function (5) and of the determinant of J (10) allows to do analysis of the separate geometrical parameters influence on the MS features.

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Кинематический и силовой анализ пятизвенной манипуляционной системы при помощи четырех пространств матрицы Якоби

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(Р е з ю м е)

В работе представлена пятизвенная равнинная манипуляционная система закроенного типа. Используется матрица имени Якоби и аналитическое решение двух задач кинематики. Приведены примеры для связи и зависимость сингулярных конфигураций манипуляционных роботов. Сделани выводы, касающие пространства матрицы и их определители.