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An Interactive Algorithm of the Multicriteria Linear Integer Programming

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1. Introduction

The interactive algorithms are widely used for solving multicriteria linear programming problems [1, 9, 15, 17]. The quality of an interactive algorithm depends to a large extent on the quality of the dialogue with the decision maker (DM). The quality of the dialogue with the DM is connected with:

– the type of information required from the DM to improve the local preferred nondominated solution;

– the time for solving the scalarizing problem;

– the type and the number of the new solutions compared with the local preferred solution;

– the possibilities for change of the strategies searching for new solutions;

– the possibilities to help the DM learn about the multicriteria problem solved.

When solving multicriteria linear programming problems (MCLP), linear programming problems are used as scalarizing problems. These are easy to solve. That is why in the interactive algorithms solving MCLP, the time needed to solve the scalarizing problems does not play a significant role. The interactive algorithms are also often used [2] to solve multicriteria linear integer programming problems (MCIP). The most of them [3, 6, 12] are modifications of interactive approaches solving MCLP that include the integrality constraints. Linear integer programming problems are used as scalarizing problems in these interactive algorithms. These problems are NP-difficult problems [4]. Moreover, finding a feasible integer solution can be as difficult as finding an optimal solution. That is why in the interactive algorithms solving MCIP the time to solve the scalarizing problem plays a significant role. For this reason an effort is made to reduce the number of the integer problems solved: approximate algorithms are used to solve the integer problems, or a possibility is provided to interrupt the exact algorithms in solving these problems; continuous problems (instead of integer problems) are solved and continuous (weak) nondominated solutions obtained are presented to the DM for evaluation (especially in the DM's learning phase). Some of the interactive algorithms work with the aspiration levels of the criteria, others use weight to denote the relative significance of the criteria or trade-off between the criteria. Many show one while others show several (weak) nondominated solutions to the DM for evaluation at each iteration.

We propose a learning-oriented [5] interactive algorithm. The main features of the algorithm proposed, are:

– They reduce the number of the integer problems solved because in most of the iterations the solutions of single criterion linear problems with continuous variables (which are easy to solve) are presented to the DM for evaluation. This is used under the assumption [12] that the criteria values for the scalarizing problems with continuous variables differ relatively little from the solutions with integer variables and under the assumption that the DM prefers to work in the criteria rather than in the variable space.;

– At every iteration the DM provides his/her local preferences in terms of the desired changes in the criteria values of some of the criteria, the desired directions of change of the other criteria and directions of the eventual deterioration of the remaining criteria, instead of aspiration levels of the criteria. The current preferred solution and the local preferences of the DM define a reference neighborhood in which the next preferred solution is searched for;

– At every iteration in a reference neighbourhood a set of continuous (weak) nondominated solutions or a set of integer near (weak) nondomnated solutions or integer (weak) nondominated solution is searched for solving continuous or integer scalarizing problems .

The multicriteria linear integer programming (I) can be formulated as:

(1) $$\text{“max”}\{f_k(x),\ k\in K\}$$

subject to:

(2) $$\sum_{j\in N} a_{ij} x_j \le b_i,\ i \in M,$$

(3) $$0 \le x_j \le d_j,\ j \in N,$$

(4) $$x_j - \text{integer},\ j \in N,$$

where the symbol “max” means that all the objective functions are to be simultaneously maximized; $K = \{1, 2, \ldots, p\}$, $M = (1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$ denote the index sets of the objective functions (criteria), the linear constraints, and the decision variables, respectively: $f_k(x)$, $k\in K$ are linear criteria (objective functions); $f_k(x) = \sum_{j\in N} c^k_j x_j$ and $x = (x_1, x_2, \ldots, x_j, \ldots, x_n)^T$ is the vector of the decision variables.

The constraints (2)–(4) define the feasible region X_1 for the integer variables.

The problem (1)–(3) is a multicriteria linear programming problem (P). The feasible region for the continuous variables is denoted by X_2. Problem (P) is a relaxation of (I).

For clarity of the exposition, we introduce a few definitions:

Definition 1. A near (weak) nondominated solution is a feasible solution in the criteria space located comparatively close to the (weak) nondominated solutions.

Definition 2. A current preferred solution is a near (weak) nondominated solution or (weak) nondominated solution chosen by the DM at the current iteration. The most preferred solution is a preferred solution that satisfies the DM to the greatest degree.

Definition 3. Desired changes of the criteria values are the amounts by which the DM wishes to increase the criteria in comparison with their value in the current preferred solution.

Definition 4. The desired directions of change of the criteria are the directions, in which the DM wishes to improve the criteria in comparison with their values at the current preferred solution.

Definition 5. Reference neighbourhood is defined by the current preferred solution; the desired changes in the values of some of the criteria, the desired directions of change of the other criteria and directions of the eventual deterioration of the remaining criteria as specified by the DM.

2. Scalarizing problems

We formulate the scalarazing problems [1, 12] under the assumption that the set of criteria K can be divided into three subsets – K_1, K_2 and K_3. The set K_1 contains the indices $k\in K$ of the criteria for which the DM wants to improve their values compared to the values in the current preferred solution. The set K_2 includes the indices $k\in K$ of the criteria for which the DM agrees to worsen their values not setting the exact values of deterioration. The set K_3 contains the indices $k\in K$ of the criteria whose values the DM wants to preserve. The set K_1 is divided into two subsets – K_1' and K_1''; K_1' contains indices of the criteria $k\in K_1$ that the DM wants to improve by desired values Δ_k, and K_1'' consists of indices of the criteria, that the DM wants to improve and for which he/she is not able to set the exact values of improving.

The following scalarizing problem, named E_1, is proposed to obtain a (weak) nondominated solution of the multicriteria integer problem (I) in the reference neighbourhood of the current preferred solution.

Minimize

$$S(x) = \max\Big[\max_{k\in K_1'}(\bar{f}_k - f_k(k))/|f_k'|,\ \max_{k\in K_2}(f_k - f_k(x))/|f_k'|\Big] + \tag{5}$$

$$+ \max_{k\in K_1''}(f_k - f_k(x))/|f_k'|,$$

subject to:

$$f_k(x) \ge \bar{f}_k,\ k\in K_3\cup K_1'', \tag{6}$$

$$x\in X_1 \tag{7}$$

where f_k is the value of the criterion with an index $k\in K$ in the current preferred solution, $\bar{f}_k = f_k+\Delta_k$ is the desired level of the criterion with an index $k\in K_1'$; f_k' – a scaling coefficient,

$$f_k' = \begin{cases} f_k \text{ if } f_k\neq 0,\\ 1, \text{ if } f_k=0.\end{cases}$$

Theorem 1. The optimal solution of the scalarizing problem E_1 is a weak efficient solution of the multicriteria integer programming problem (I).

For a proof, please see the Appendix.

Consequence. Theorem 1 is true for arbitrary values of f_k, $k\in K$.

The proof of this consequence follows from the fact that the proof of Theorem 1 does not assume any constraints on the values of the criteria f_k, $k\in K$.

To obtain a (weak) nondominated solution for the problem (P) in the reference neighbourhood of the current preferred solution, we may use the scalarizing problem E_2, which is obtained from E_1 replacing constraint (7) by constraint:

(8) $$x \in X_2 .$$

Theorem 2. The optimal solution of the scalarizing problem E_2 is a weak efficient solution of the multiple criteria linear problem (P).

The proof of Theorem 2 is analogous to the proof of Theorem 1 because nature of the variables x^*_i, $i = \overline{1, n}$, is not explicitly used.

Because the objective function of the scalarizing problem E_1 is nondifferentiable, one may solve the following equivalent mixed integer programming E_1':

(9) $$\min\ (\alpha+\beta)$$

subject to:

(10) $$\alpha \geq (\bar{f}_k - f_k(x))/\,|\,f_k'|,\ k \in K_1',$$

(11) $$\alpha \geq (f_k - f_k(x))/\,|\,f_k'|,\ k \in K_2,$$

(12) $$\beta \geq (f_k - f_k(x))/\,|\,f_k'|,\ k \in K_1'',$$

(13) $$f_k(x) \geq \bar{f}_k,\ k \in K_1'' \cup K_3,$$

(14) $$x \in X_1,$$

(15) $$\alpha, \beta - \text{arbitrary}.$$

Problems E_1 and E_1' have the same feasible sets of the variables. The value of the objective functions of problems E_1 and E_1' are equal. This follows from the following assertion:

The scalarizing problem E_1' has four properties, that help to improve the dialogue with the DM, as with respect to the required from him/her information and with respect to the reducing of the waiting time for evaluation of new solutions also. The first property is connected with the required information from the DM. Instead of the aspiration levels of every criteria for the defining of the reference point [9, 12, 17], the DM has to provide only changes in the criteria values of some of the criteria and the directions of change of the another criteria to specify the reference neighbourhood. The second property is that the current preferred solution is an initial feasible solution of the next integer problem E_1'. This facilitates the single criterion algorithms, especially the heuristic algorithms. The third property is that the feasible solutions of problem E_1' are near to the nondominated surface of the multicriteria integer problem (I). The application of heuristic algorithms to solve problem E_1' will lead to near (weak) nondominated solutions quickly, thus reducing the waiting time for the dialogue with the DM. The comparatively quick finding of more solutions for evaluation by the DM is important during the learning phase of the DM. The forth property of the problem E_1' is that with it the DM can realize the search strategy "no great benefit - little loss". The solutions obtained in the reference neighbourhood are comparatively close, which makes it easier for the DM to compare several solutions and choose the next preferred solution.

The scalarizing problem (E_2) is equivalent to the following linear programming problem E_2':

(16) $$\min\ (\alpha+\beta)$$

subject to:

(17) $$\alpha \geq (\bar{f}_k - f_k(x))/|f_k'|,\ k \in K_1',$$
(18) $$\alpha \geq (f_k - f_k(x))/|f_k'|,\ k \in K_2,$$
(19) $$\beta \geq (f_k - f_k(x))/|f_k'|,\ k \in K_1'',$$
(20) $$f_k(x) \geq \bar{f}_k,\ k \in K_1'' \cup K_3,$$
(21) $$x \in X_2,$$
(22) $$\alpha, \beta - \text{arbitrary}.$$

The parametric extension of the scalarizing problem E_2' (denoted by $\bar{E}_2'$) has the following form (similar to the one in [9])

(23) $$\min\ (\alpha+\beta)$$

subject to:

(24) $$f_k(x) + |f_k'|\alpha \geq f_k + (\bar{f}_k - f_k)\ t,\ k \in K_1',$$
(25) $$f_k(x) + |f_k'|\alpha \geq f_k - t,\ k \in K_2,$$
(26) $$f_k(x) + |f_k'|\beta \geq f_k + t,\ k \in K_1'',$$
(27) $$f_k(x) \geq \bar{f}_k,\ k \in K_1'' \cup K_3,$$
(28) $$x \in X_2,$$
(29) $$t \geq 0,$$
(30) $$\alpha, \beta - \text{arbitrary}.$$

Problems E_2' and $\bar{E}_2'$ have the same properties as problem E_1', but they give continuous solutions.

Let us assume that we have found a (weak) nondominated solution of problem (P) with the help of the scalarizing problems E_2' and $\bar{E}_2'$ and wish to find a (weak) nondominated solution of problem (I), which is near the (weak) nondominated solution of problem (P). Let us denote by $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_p)^T$ a (weak) nondominated solution of problem (P).

To find a (weak) nondominated solution of problem (I), close to the (weak) nondominated solution $\hat{f}_k$ of problem (P), the following Chebychev's problem E_3 may be used [26]:

Minimize

(31) $$S(x) = \max_{k \in K} (\hat{f}_k - f_k(x))/|\hat{f}_k'|,$$

subject to:

(32) $$x \in X_1,$$

where

$$\hat{f}_k' = \begin{cases} \hat{f}_k, & \text{if } \hat{f}_k \neq 0, \\ 1, & \text{if } \hat{f}_k = 0. \end{cases}$$

This problem is equivalent to the following mixed integer programming problem E_3':

(33) $$\min\ \alpha$$

under the constraints:

(34) $$\alpha \geq (\hat{f}_k - f_k(x))/|\hat{f}_k'|,$$
(35) $$x \in X_1,$$
(36) $$\alpha - \text{arbitrary}.$$

3. General scheme of the algorithm

A learning-oriented interactive algorithm solving multicriteria linear integer problems can be suggested on the basis of the scalarizing problems E_1', E_2', $\bar{E}_2'$ and E_3'. The dialogue with the DM has been improved with respect to the information required from him/her; to the time when he/she is expecting a new solution; to the possibility for evaluation of more new solutions and to the learning possibilities of the specifics of the problem solved.

The basic steps of the algorithm are the following:

Step 1. An initial (weak) nondominated solution of the multicriteria problem (P) is defined, setting $f_k = 1$, $k \in K$, $\bar{f}_k = 2$, $k \in K$, and solving problem E_2'.

Step 2. Ask the DM to specify the reference neighbourhood of the current preferred solution defining desired changes in the values of some criteria, desired directions of change of other criteria and the directions of the eventual deterioration of remaining criteria.

Step 3. Ask the DM to define whether to search for a (weak) nondominated solution of the multicriteria problem (P) or near (weak) nondominated solutions of the multicriteria problem (I). In the first case, *Step 4* is executed, in the second case go to *Step 6*.

Step 4. Ask the DM to specify parameter s – the maximal number of (weak) nondominated solutions of the multicriteria problem (P) which can be saved in the set M_1. Solve the scalarizing problem $\bar{E}_2'$ with the help of an algorithm of linear parametric programming. Present the set M_1 to the DM for evaluation and selection. In case the DM wants to see a (weak) nondominated solution of the multicriteria problem (I), close to the current preferred solution of the multicriteria problem (P), *Step 5* is executed, otherwise – *Step 2*.

Step 5. Solve problem E_3'. Show the (weak) nondominated solution of multicriteria problem (I) obtained by the exact integer algorithm chosen for solving problem, or a near (weak) nondominated solution of the multicriteria problem (I) obtained by the heuristic integer algorithm. If the DM approves this solution as current preferred solution of the multicriteria problem (I) go to *Step 2*. If this solution is the last preferred solution – *Stop*.

Step 6. Ask the DM to choose the type of the algorithm – exact or heuristic. If the DM selects an exact algorithm – go to *Step 8*.

Step7. Ask the DM to specify s – the maximal number of near (weak) nondominated solutions of the multicriteria problem (I), which can be stored in the set M_1. Solve the scalarizing problem with the help of an heuristic integer algorithm and present the set M_1 to the DM for evaluation and selection the current preferred solution of the multicriteria problem (I). If the current preferred solution is the last preferred solution – *Stop*, other wise – go to *Step 2*.

Step 8. Solve problem. Show the (weak) nondominated solution or near (weak) nondominated solution (if the computing process is interrupted) of the multicriteria problem (I) to the DM. In case the DM approves this solution as a current preferred solution of the multicriteria problem (I) go to *Step 2*. If the solution is the last preferred solution – *Stop.*

The proposed algorithm for solving multicriteria linear integer problems is a learning oriented [7] interactive algorithm and the DM controls the dialogue, the computations and the stopping conditions.

Problems of linear parametric programming (scalarizing problems $\bar{E}_2'$) are solved in the interactive algorithm. The linear parametric programming problems are easily solved problems and the DM must not wait long for the obtaining and estimation of new solutions. Problems of mixed integer linear programming (scalarizing problems E_1' and E_3') are also solved. The number of the integer problems solved can be very small. They are solved only in the cases when the DM feels uncomfortable to operate with continuous variables or when he is searching for an integer solution near to the current preferred continuous solution. In the first case it is appropriate (especially in the learning process) to solve the integer problems with the help of approximate algorithms. The use of approximate algorithms [7, 14, 16] operating efficiently in a "narrow feasible area" and a known initial feasible integer solution enables the finding of good and in many cases – optimal solutions of the problems E_1'. The evaluation of more than one, even they be approximate (weak) nondominated solutions, enable the DM to learn faster with respect to the problems being solved.

The DM operates mainly in the criteria space, because in most of the cases the criteria have physical or economic interpretation and this enables the more realistic estimation and choice. The information required from the DM refers only to the defining of a reference neighbourhood of the current preferred solution and sometimes, if he/she wants, to the presenting of inter- and intra-criteria information.

4. Conclusion

A learning-oriented interactive algorithm is proposed based on the reference neighbourhood approach to solve multicriteria linear integer programming problems. This algorithm provide the opportunity to improve the dialogue with the DM with respect to several features:

– according to DM's wish, he/she may set different type and different quantity of information at each iteration;

– the time during which he/she is expecting solutions for evaluation and choice is reduced;

– his/her possibilities for learning the specifics of the multiple criteria integer problems being solved can be increased.

These features of the proposed interactive algorithm characterise it as an appropriate and user-friendly algorithm solving multicriteria linear integer programming problems.

Appendix

Theorem 1. The optimal solution of the scalarizing problem is a weak efficient solution of the multicriteria integer programming problem (I).

P r o o f. Let K_1' and $K_1'' \neq \varnothing$.

Let x^* be an optimal solution of problem E_1. Then the following condition is satisfied:

$$S(x^*) \leq S(x),\ x \in X,$$

and $f_k(x) \geq \bar{f}_k,\ k \in K_1'' \cup K_3$,

Let us assume that x^* is not a weak Pareto optimal solution of the initial multiple criteria integer problem (I). In this case there must exist $x' \in X$, for which:

$$f_k(x') > f_k(x^*) \text{ for } k \in K \text{ and } f_k(x^*) \geq \bar{f}_k,\ k \in K_1'' \cup K_3\ . \tag{37}$$

After transformation of the objective function $S(x)$ of the scalarizing problem E_1, using the inequalities (37), the following relation is obtained:

$$S(x') = \max\,[\max_{k\in K_1'} (\bar{f}_k - f_k(x'))/\,|\,f_k'|,\ \max_{k\in K_2}(f_k - f_k(x'))/\,|\,f_k'|\,] +$$

$$+ \max_{k\in K_1''} (f_k - f_k(x'))/\,|\,f_k'| =$$

$$= \max\,[\max_{k\in K_1'} (\bar{f}_k - f_k(x^*) + (f_k(x^*) - f_k(x'))/\,|\,f_k'|\,]\ ,$$

$$\max_{k\in K_2} (f_k - f_k(x^*) + (f_k(x^*) - f_k(x'))/\,|\,f_k'|\,] + \tag{38}$$

$$+ \max_{k\in K_1'} (f_k - f_k(x^*) + (f_k(x^*) - f_k(x'))/\,|\,f_k'|\,] <$$

$$< \max\,[\max_{k\in K_1''} (\bar{f}_k - f_k(x^*))/\,|\,f_k'|,\ \max_{k\in K_2}(f_k - f_k(x^*))/\,|\,f_k'|\,] +$$

$$+ \max_{k\in K_1'} (f_k - f_k(x^*))/\,|\,f_k'| = S(x^*).$$

It follows from (38) that $S(x') < S(x^*)$ and $f_k(x^*) \geq \tilde{f}_k$, $k\in K_1'' \cup K_3$, which contradicts to (37). Hence x^* is a weak efficient solution of the multiple criteria integer problem (I).

References

1. Benayoun, R., J. de Montgolfier, J. Tergny, O. Larichev. Linear programming with multiple objective functions: Step Method (STEM). —Mathematical Programming, 1971, No 1, 366-375.
2. Climaco, J., G. Ferreira, M. E. Captivo. Multicriteria integer programming: An overview of the different algorithmic approaches. – In: Multicriteria Analysis (J. Climaco, Ed.), Springer, 1997, 248-258.
3. Gabbani, D., M. Magazine. An interactive heuristic approach for multiobjective integer program. – Journal of Operational Research Society, **37**, 1986, 285-291.
4. Garey, M., D. Johnson. Computers and Intractability: A Guide to the Theory of NP-completeness. San Francisko, W. H. Freeman, 1979.
5. Gardiner, L. R., D. Vanderpooten. Interactive multiple criteria procedures: Some reflections. – In: Multicriteria Analysis (J. Climako, Ed.), Springer, 1997, 290-301.
6. Hajela, P., C. J. Shih. Multiobjective optimum design in mixed integer and discrete design variable problems. – AIAA Journal, **28**, 1990, 670-675.
7. Ibaraki, T., T. Ohashi, H. A. Mine. A heuristic algorithm for mixed integer programming problems. – Mathematical Programming Study, **2**, 1974, 115-136.
8. Karwan, M. H., S. Zionts, B. Villareal, R. Ramesh. An improved interactive multicriteria integer programming algorithm. – In: Decision Making with Multiple Objectives (Y. Y. Haimes and V. Chankong, Eds.), Springer-Verlag, 1985, 261-271.
9. Korhonen, P., J. A. Laakso. Visual interactive method for solving the multiple criteria problem. – European Journal of Operational Research, **24**, 1986, 277-287.
10. Marcotte, O., R. Soland. An interactive branch and bound algorithm for multicriteria optimisation. – Management Science, **32**, 1986, 61-75.
11. Murtagh, B. A. Advanced Linear Programming: Computation and Practice. New York, Mc Graw-Hill, 1981.
12. Narula, S. C., V. Vassilev. An interactive algorithm for solving multiple objective integer linear programming problems. – European Journal of Operational Research, **79**, 1994, 443-450.
13. Nemhauser, G. L.. L. A. Wolsey. Integer and Combinatorial Optimization. New York, Wiley, 1988.

14. R e e v e s, C. R. Modern Heuristic Techniques for Combinatorial Problems. Oxford, Blackwell Scientific Publications, 1993.
15. S t e u e r, R. An interactive multiple objective linear programming procedure. – In: TIMS Studies in the Management Sciences (M. K. Star and M. Zeleny, Eds.), North Holland, **6**, 1977, 225-339.
16. V a s s i l e v, V., K. G e n o v a. An algorithm of integer feasible directions for linear integer programming. – European Journal of Operational Research, **52**, 1991, 203-214.
17. W i e r z b i c k i, A. P. The use of reference objectives in multiobjective optimization. – In: Multiple Criteria Decision Making: Theory and Application. Lecture Notes in Economics and Mathematical Systems (G. Fandel and T. Gal, Eds.), Springer Verlag, **177**, 1980, 468-486.

Интерактивный алгоритм для решения многокритериальных линейных целочисленых задач

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(Р е з ю м е)

Предлагается ориентированный к обучении интерактивный алгоритм отправной области для решения задач многокритериального линейного целочисленного программирования. Лицо, принимащющее решение (ЛПР), задает свои локальные предпочитания как желанные перемены стоимостей некоторых критериев, желанные направления в перемене стоимостей других критериев и направления евентуального ухудшения стоимостей части или всех остальных критериев. На их основе формируются два типа скаляризирующих функций, при помощи которых на каждой итерации определяются одно или больше целочисленные или непрерывные (слабо) недоминированные решения.

Предложенный алгоритм дает возможность ЛПР изменять свои стратегии поиска, использовать и непрерывные недоминированные решения (для сокращения времени поиска), обучаться быстрее в специфике решаемой многокритериальной задачи.