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A Network Flow with Additional Linear Equalities*

Vassil Sgurev

Institute of Information Technologies, 1113 Sofia

1. Introduction

The wide distribution of the methods and software tools based on network flows is the reason and explanation for the continuous theoretical interest towards them and their practical applicability.

The classical flows of F o r d and F u l k e r s o n [1, 2, 3] are most widely spread, the capacity in them being defined by upper and lower bounds of the arc flow function.

The different variants, in which besides this method of defining the capacity, some linear inequalities on sets of arcs are applied, have been discussed in [4, 5].

In [6, 7] a more general class of network flows is proposed, when the arcs capacities are replaced by linear inequalities of the flow function on arcs subsets with arbitrary coefficients in these inequalities. The class of network flows obtained is called a linear flow.

A new class of network flows is defined in [8], in which the capacity of the separate arcs is replaced by one linear equality with arbitrary coefficients, in which all the arc flow functions are included as variables. This class of flows is called a network flow with one linear equality, or OLE-flow.

In the present paper the previous OLE-flow is generalized for the case when the arc flow functions are enclosed not by one, but by many linear equalities with arbitrary coefficients. Main attention is paid to the problems of existence of such a class of network flows.

2. Definition of a network flow with additional linear equalities

Let a graph $G(N, U)$ be given with a set of nodes N and a set of arcs U. The elements of the set M are the indices of all the simple oriented paths from the source s towards the sink t, in which there are no cycles and in any path $\mu \in M$ the separate arcs and nodes

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are included only once [1, 2]. As we are going to use further on only simple oriented paths, they will be called just paths.

The set of all the arcs on the path with an index $\mu \in M$ will be denoted by $U(\mu)$. It is assumed that for the graph $G(N, U)$, the following is valid:

(1) $$U = \bigcup_{\mu \in M} U(\mu).$$

Let a network flow be defined on the graph $G(N, U)$ by the following constraints: for each $x \in N$

(2) $$f(x, N) - f(N, x) = \begin{cases} v, \text{ if } x = s, \\ 0, \text{ if } x \neq s, t, \\ -v, \text{ if } x = t; \end{cases}$$

(3) $$\sum_{(x, y) \in D_i} b_i(x, y)\, f(x, y) = C_i,\ i \in I,$$

(4) $$f(x, y) \geq 0;\ (x, y) \in U,$$

where I is a set of the indices of the linear equalities; $C_i \geq 0$ are rational nonnegative integers; $D_i \in U$ are the subsets from the division of U, i.e., for each $i, j \in I$, the following is valid

(5) $$D_i \cap D_j = \varnothing;\ \bigcup_{i \in I} D_i = U;$$

$\varnothing$ is an empty set, v and f – a flow and an arc flow function such that

(6) $$v \geq 0;\ f(x, y) \geq 0;\ (x, y) \in U;$$

(7) $$f(x, N) = \sum_{y \in \Gamma^1(x)} f(x, y);\ f(N, x) = \sum_{y \in \Gamma^{-1}(x)} f(y, x),$$

R' is a set of all the rational nonzero numbers; $\Gamma^1(x)$ and $\Gamma^{-1}(x)$ – an image and an inverse image of x into N.

It is assumed that for the graph $G(N, U)$

(8) $$(N, S) = (t, N) = \varnothing;$$

(9) $$b_i(x, y) \begin{cases} \in R', \text{ if } (x, y) \in D_i, i \in I; \\ = 0 \text{ otherwise.} \end{cases}$$

The network flow, defined by relations (2) up to (4) will be called a flow with additional linear equalities or an ALE-flow.

We shall make the basic assumption that each subset D_i, $i \in I$, is contained in the set of arcs (1) of all the simple oriented paths from s towards t, i.e.,

(10) $$\bigcup_{\mu \in M} U(\mu) \cap D_i = D_i.$$

The following denotations are introduced:

$M_i \subseteq M$ is a set of these paths from M, in which at least one arc from D_i is included, i.e.

$$M_i = \{\mu | \mu \in M;\ U(\mu) \cap D_i \neq \varnothing\};\ i \in I.$$

$U_i(\mu)$ is a set of arcs of the path μ, enclosed in D_i, i. e.

(11) $$U_i(\mu) = U(\mu) \cap D_i;\ i \in I,\ \mu \in M_i;$$

U_i is a set of all the arcs on all the paths from M_i, i. e.

(12) $$U_i = \bigcup_{\mu \in M_i} U(\mu)\ ;\ i \in I;$$

$B_i(\mu)$ is the sum of the coefficients, corresponding to $U_i(\mu)$, at that for each $i \in I$ and $\mu \in M_i$

(13) $$B_i(\mu) = \begin{cases} \sum_{(x,y)\in U_i(\mu)} b_i(x,y), & \text{if } U_i(\mu) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

$v(\mu)$ will denote this part of the flow v, which corresponds to the path μ. The following transition from a network flow v in an arcs-nodes form towards a flow $v(h)$ in arcs-paths form can be realized with the help of condition (8) and Theorem 2.2 from [1]:

(14) $$v(h) = \sum_{\mu \in M} v(\mu) = v.$$

We shall use further on the denotation v only.

The following coefficients will be defined:

(15) $$\alpha_\mu = \begin{cases} v(\mu)/v, & \text{if } v>0;\ \mu \in M; \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the definition above given that for each $\mu \in M$ and $v>0$

(16) $$\sum_{\mu \in M} \alpha_\mu = 1; \quad 0 \le \alpha_\mu \le 1.$$

The above relations mean that the flow v can be divided among the paths with indices from the sets M. The requirements for the preserving of equations (2) are satisfied at that, i.e., for each $(x, y) \in U(\mu)$, $\mu \in M$, it can be written that $f(x,y) = v(\mu)$.

Let the number of paths in M be equal to m, i.e., $|M| = m$. In case the paths indices are numbered in a fixed manner, $\{\alpha_\mu\}$ can be regarded as elements of the vector α,

(17) $$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m).$$

Definition 1. The value

(18) $$B(\alpha, i) = \sum_{\mu \in M_i} \alpha_\mu B_i(\mu),\quad i \in I,$$

will be called an α_i-factor.

According to (13) $\{B_i(\mu)\}$ will depend on apriori given coefficients $\{b_i(x, y)\}$ only and hence α_i-factor is a function of the elements of the vector (17).

Definition 2. Each realization of the vector α, corresponding to requirements (15) and (16), will be called an α-realization.

It follows from (15) and (16) that depending on the values of v each α-realization can be assigned a set of flow realizations $\{v(\mu) | \mu \in M\}$.

The following three confirmations follow directly from relations (13) up to (18).

Confirmation 1. Only one α_i-factor corresponds to each α-realization and index $i \in I$.

Confirmation 2. α_i-factor is invariant with respect to the value of the flow v and it depends on the α-realization and the value of the parameters $B_i(\mu)$; $\mu \in M_i$.

Confirmation 3. There exists biunique correspondence between each α-realization and the corresponding values of the flow $\{v(\mu) / \mu \in M\}$ at a fixed value of v.

3. Conditions for the existence of a network flow with additional linear equalities

Lemma 1. In the case of an ALE-flow for each $i \in I$

(19) $$\sum_{(x,y)\in D_i} b_i(x,y)\, f(x,y) = \sum_{\mu\in M_i} v(\mu) B_i(\mu).$$

Proof. Taking in mind [1], the following coefficients will be introduced:

(20) $$\alpha_\mu(x,y) = \begin{cases} 1, \text{ if } (x,y)\in U_i(\mu); \\ 0 \text{ otherwise.} \end{cases}$$

When passing from the flow in arcs-nodes form towards a flow in arcs-paths form [1], for each $(x,y)\in U_i(\mu)$, $\mu\in M_i$, it will be written

(21) $$f(x,y) = \sum_{\mu\in M_i} \alpha_\mu(x,y)\, v(\mu).$$

On the basis of (14), (19) and (20), the left side of (19) can be represented in the following way:

(22) $$\sum_{(x,y)\in D_i} b_i(x,y)\, f(x,y) = \sum_{(x,y)\in D_i} b_i(x,y) \sum_{\mu\in M_i} \alpha_\mu(x,y)\, v(\mu) =$$
$$= \sum_{\mu\in M_i} v(\mu) \sum_{(x,y)\in D_i} \alpha_\mu(x,y)\, b_i(x,y) = \sum_{\mu\in M_i} v(\mu) \sum_{(x,y)\in v_i(\mu)} b_i(x,y).$$

The equalities above given and (18) prove (19) ■

Consequence 2. There exists a relation

(22) $$\sum_{(x,y)\in D_i} b_i(x,y)\, f(x,y) = vB(\alpha, i).$$

The equality above given follows directly from relations (14) upto (19) ■

The following results define the conditions for the existence of an ALE-flow.

Theorem 3. If for any $i\in I$

(24) $$C_i > 0 \text{ and } B_i(\mu) \le 0 \text{ for each } \mu\in M_i,$$

then there does not exist any ALE-flow, for which all conditions from (2) upto (4) are satisfied.

Proof. From the second requirement in (24) and the non-negativeness of $\{v(\mu) | \mu\in M_i\}$ it follows that

$$\sum_{\mu\in M_i} v(\mu)\, B_i(\mu) \le 0.$$

The above inequality and relations (3) and (19) lead to the requirement $C_i \le 0$, which contradicts to the first of the inequalities in (24) and proves the impossibility for existence of an ALE-flow under conditions (24) ■

Lemma 4. If $C_i > 0$, $i\in I$, then the requirement

(25) $$B_i(\mu) > 0,\ i\in I, \text{ for at least one } \mu\in M_i,$$

is a necessary, but not sufficient condition for the existence of an ALE-flow from (2) upto (4).

Proof. Necessity. It is assumed that requirement (25) is not satisfied, i.e., that all the values $\{B_i(\mu) | i \in I; \mu \in M_i\}$ are nonpositive. Then from the nonnegativeness of $\{v(\mu) | \mu \in M\}$ and relation (19) the violating of (3) for every $i \in I$ follows. This is a contradiction.

Insufficiency. Let (25) be satisfied, and the graph $G(N, U)$ be defined so that

$$\{\mu'\} = M; \{i, j\} = I; U_i(\mu') = \{(x, y)\}; U_j(\mu') = \{(y, z)\};$$

$$B_i(\mu') = b_i(x, y) > B_j(\mu') = b_j(y, z) > 0; C_i = C_j > 0.$$

It follows from the relations above given and (3) and (19) that

$$v\{\mu'\} = \frac{C_i}{B_i(\mu')} = \frac{C_j}{B_j(\mu')};$$

which is not possible, since in the last equality the numerators coincide, but the denominators – no. Hence, (25) is not a sufficient condition for the existence of the ALE-flow▫

Lemma 5. If $C_i = 0$, $i \in I$, then the requirement

(26) $$B_i(\mu') = 0 \text{ for each } i \in I \text{ and } \mu \in M_i$$

is a sufficient, but not a necessary condition for the existence of a nonzero ALE-flow.

Proof. Sufficiency. It follows from relation (19) that the equalities (26) lead to the satisfying of each of the requirements from (2) upto (4), and the flow v may have a nonzero value.

Nonnecessity. Let the graph $G(N, U)$ be defined as follows:

$$\{\mu', \mu''\} = M; \{i, j\} = I; U_i(\mu') = \{(x, y)\}; U_i(\mu'') = \{(x, z)\};$$

$$U_j(\mu'') = \{(z, y)\}; B_i(\mu') = b_i(x, y) > 0, B_i(\mu'') = b_i(x, z) < 0;$$

$$B_j(\mu'') = b_j(z, y) = 0.$$

From the relations above given and (19) it follows that in case the nonzero flows $v\{\mu'\}$ and $v\{\mu''\}$ are arranged in such a way that

$$v(\mu')B_i(\mu') - v(\mu'')B_i(\mu'') = 0,$$

then all the requirements from (2) upto (4) will be satisfied, and (26) will not be valid for the constraint (3) with an index i. Hence, it is not a necessary condition for the existence of a nonzero ALE-flow▫

Lemma 6. If

(27) $$C_i = 0 \text{ and } B_i(\mu) \neq 0, \text{ for each } i \in I \text{ and } \mu \in M_i,$$

then the requirement that for each $i \in I$ there should be at least two paths $\mu' \in M_i$ and $\mu'' \in M_i$ such that

(28) $$B_i(\mu') > 0 \text{ and } B_i(\mu'') < 0$$

is a necessary but not sufficient condition for the existence of a nonzero ALE-flow.

Proof. Necessity. It is assumed that there exists at least one $i \in I$ for which (28) is not satisfied and for it

(29) $$\sum_{\mu \in M_i} v(\mu) > 0 \text{ and } B_i(\mu) > 0 \text{ for each } \mu \in M_i, \text{ or}$$

(30) $$\sum_{\mu \in M_i} v(\mu) > 0 \text{ and } B_i(\mu) < 0 \text{ for each } \mu \in M_i.$$

From (19) and (29) it follows that for equality (3) with an index i, the coefficient C_i is positive. Under the assumptions (19) and (30) the same coefficient has a negative value. In both cases this contradicts to assumption (27) and proves the necessity of (28).

Insufficiency. Let the graph $G(N, U)$ be defined in the following way:

(31) $\{\mu', \mu''\} = M$; $\{i, j\} = I$; $U_i(\mu') = \{(x, y)\}$; $U_i(\mu'') = \{(x, z)\}$;

(32) $U_j(\mu') = \{(y, t)\}$; $U_j(\mu'') = \{(z, t)\}$; $B_i(\mu') = b_i(x,y) > 0$;

(33) $B_i(\mu'') = b_i(x, z) < 0$; $B_j(\mu') = b_j(y, t) > 0$; $B_j(\mu'') = b_j(z, t) < 0$.

From (3), (19) and the above relations it follows that the satisfying of condition (27) leads to:

(34) $$\frac{v(\mu')}{v(\mu'')} = \frac{|B_i(\mu'')|}{|B_i(\mu')|} = \frac{|B_j(\mu'')|}{|B_j(\mu')|};$$

where $|B_i(\mu'')|$ is the absolute value of $B_i(\mu'')$ respectively.

If $|B_i(\mu'')| = |B_i(\mu')|$ and $|B_j(\mu')| > |B_j(\mu'')|$ then the comparison of the above relations with (3), (19) and (27) leads to the impossibility for (3) at an index $j \in I$. Besides this according to (32) and (33) inequalities (28) are satisfied. The contradiction obtained shows that (28) is not a sufficient condition for the existence of a nonzero ALE-flow.

Lemma 7. The necessary and sufficient condition for the existence of an ALE-flow from (2) upto (4) is the presence of such a realization of the flow (17), for which

(35) $$vB(\alpha, i) = C_i \text{ for each } i \in I.$$

Proof. Necessity. It is assumed that equality (34) is not satisfied for any $i \in I$, i.e., that

(36) $$vB(\alpha, i) > C_i;\ i \in I.$$

From (23) and (36) it follows that for the same $i \in I$

$$\sum_{(x,y)\in D_i} b_i(x,y)\, f(x,y) > C_i,$$

which contradicts to (3) and proves the impossibility of (36) for each $i \in I$. In an analogous way it can be shown that the assumption

$$vB(\alpha, i) < C_i \text{ for an arbitrary } i \in I$$

leads to a contradiction with (3). Hence, the requirement (35) is a necessary condition for the existence of an ALE-flow.

Sufficiency. Let for every $i \in I$ the equality (35) be satisfied. Then from (23) and (35) it follows that for each $i \in I$ equalities (3) are valid.

Lemma 8. If for every possible realization of the flow (17) at least one index $i \in I$ is found such that

(37) $$C_i > 0 \text{ and } B_i(\alpha, i) < 0;$$

then there does not exist any ALE-flow from (2) upto (4).

Proof. It follows from (3) and (23) that for each $i \in I$ it can be written:

$$vB(i, \alpha) = C_i.$$

Since by definition

$$v \geq 0,$$

then the correspondence of the last two relations with (37) leads to a contradiction and proves the impossibility for ALE-flow existence or the assumptions of Lemma 8.

It will be interesting to investigate the capacity of the ALE-flow, caused by the set of additional linear equalities, as well as its connection with the maximal and minimal values of this flow.

Conclusion

The results obtained are reduced in the general case to the following:

1. The classical network flow of Ford and Fulkerson does not need any proof for existence, since at any values of the capacity there exists at least one realization of the flow, corresponding to the zero values of the arc flow functions, which satisfies all the requirements for this flow.

The values of the coefficients $\{C_i\}$ and $\{b_i(x, y)\}$ of the linear equalities (3) are indicated in Theorem 3 and Lemma 8 of the present paper, for which no ALE-flow from (2) upto (4) exists.

2. A number of requirements have been obtained – necessary and sufficient from Lemma 7, necessary but not sufficient – from Lemma 4 and Lemma 6 and sufficient, but not necessary – from Lemma 5, which indicate the conditions for existence of an ALE-flow from (2) upto (4).

3. A realization and a realization factor of an ALE-flow are introduced in the form of arcs-paths and relations characterizing these notions have been obtained.

4. It is shown that for no values of the coefficients $\{C_i\}$ and $\{b_i(x,y)\}$ of an ALE-flow it can be reduced to the classical network flow.

Some conditions are given in [6, 7] when the linear flow is reduced to the classical flow of Ford and Fulkerson. In this aspect the ALE-flow and the OLE-flow have principally different properties with respect to the classes of network flows already investigated. This is connected with the fact that the ALE-flow and the OLE-flow are defined by equalities only, not using inequalities.

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Сетевой поток с дополнительными линейными равенствами

Васил Сгурев

Институт информационных технологий, 1113 София

(Р е з ю м е)

В работе вводится класс потоков в сетях, в которых не задаются пропускные способности отдельных дуг, а дуговые потоковые функции ограничиваются множеством линейных равенств с неотрицательными коэффициентами в правой стороне и произвольными коэффициентами в левой стороне. Этот поток назван сетевым потоком с дополнительными линейными равенствами, или ДЛР-потоком.

На основе определяемых понятий α-реализации потока в виде дуги-пути и α_i-фактора потока получено ряд необходимых и/или достаточных условий существования ДЛР-потока, также как и значения коэффициентов линейных равенств, при которых невозможно существование этого потока.