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A Method for Quantitaive Representation of Many-valued Logical Systems*

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1. Introduction

The use of many-valued logical systems in artificial intelligence knowledge bases is connected with their efficient representation by formal quantitative methods and models.

In 1986 C. Blair and R. Jeroslaw [1] proposed Programming Techniques for Propositional Logic [1]. Later on the latter of the authors has generalized the results in this direction in his last book [2]. In [3] a quantitative approach to logic inference is described.

In [4] a model, interpreting two-valued logic by a network flow with additional linear equalities and inequalities is suggested. The present paper proposes the application of this network-flow approach to various many-valued logical systems.

The network flows are defined on a graph $G(X, U)$, where X is the set of arcs, and U – the set of nodes [5]. The conservation equation, in which each $x \in X$, is basic for the flows:

$$\sum_{i \in I_x} f_i - \sum_{j \in J_x} f_j = \begin{cases} v, \text{ if } x = s, \\ 0, \text{ if } x \neq s, t, \\ -v, \text{ if } x = t; \end{cases} \tag{1}$$

where I_x and J_x are sets of the indices of the out- and incoming arcs for the node x, and $s \in X$ and $t \in X$ are a source and a sink respectively.

The ark flow function does not change along the arc length at network-flow interpretation of two-valued logic. This requirement cannot be satisfied in many-valued logic, that is why a generalized flow with profits and losses in which the flow function can be different at the initial and final node of the arc [5] is used further on in the models suggested. Besides, unlike the classical network flow, which is limited by an arc capacity, more general linear inequalities are used in the models proposed.

Several types of many-valued logic [6, 9] correspond to two-valued logic [8], that is why an interpretation of the logical operations disjunction, conjunction, negation and

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implication with the help of a generalized network flow in three of the most widely spread many-valued logical systems, will be described.

2. Network-flow description of the logical operations in some classes of many-valued logical systems

2.1. Many-valued logical system of J. Lucasiewicz

Historically this is the first system in which an attempt has been made to proceed from two-valued towards many-valued logic. The logical variables x, y, z in it obtain values within the interval from 0 up to 1, indicating at that the true value by 1, and the false – by 0, the remaining values between them corresponding to the other states.

In this system the disjunction z between x and y is defined as

(2) $$z = \max[x, y].$$

Fig. 1 shows a graph with three nodes, three arcs and three non-negative arc functions f_1, f_2 and f_3, for which $I_x = \{1, 2\}$ and $J_x = \{3\}$.

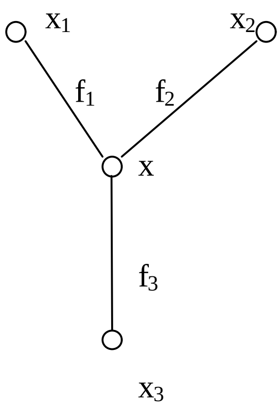

Fig. 1

Let the following relations be defined among the three functions:

(3) $$f_3 = k(f_1 - f_2) + f_3;$$

(4) $$0 \le f_1 \le f_3 \le 1;$$

(5) $$0 \le f_2 \le f_3 \le 1;$$

(6) $$k = 0 \text{ or } 1.$$

After comparison of the above equalities and inequalities, the inference is made that

(7) $$f_3 = \begin{cases} f_2, \text{ if } f_1 \le f_2;\ k = 0, \\ f_1, \text{ if } f_1 > f_2;\ k = 1, \end{cases}$$

$$\text{or } f_3 = \max[f_1, f_2]$$

which corresponds to (1) and interprets the many-valued disjunction of Lucasiewicz.

The disjunction can be represented by relations (3) and (6) and the inequalities

(8) $$0 \le f_3 \le f_1 \le 1;$$

(9) $$0 \le f_3 \le f_2 \le 1;$$

These requirements lead to

(10) $$f_3 = \min\,[f_1,\ f_2] = \begin{cases} f_1, \text{ if } f_1 \leq f_2; \\ f_1, \text{ if } f_1 > f_2. \end{cases}$$

The negations in the many-valued logical system studied are represented for f_1 and f_2 by

(11) $$f_3 = 1 - f_1 \text{ or } f_3 = 1 - f_2$$

respectively.

The implication $f_1 \to f_2$ in Lucasiewicz' system is specific when $f_2 < f_1$ and it can be illustrated by:

(12) $$f_3 = \min\,[f_1,\ f_2] = \begin{cases} f_1, \text{ if } f_1 \leq f_2; \\ f_1, \text{ if } f_1 > f_2. \end{cases}$$

The relation above given can be obtained by three functions f_1, f_2 and f_3 from Fig. 1 in the following way:

(13) $$f_3 = 1 - k\,(f_1 - f_2);$$

(14) $$0 \leq k - f_2 - f_1 < 1;$$

(15) $$0 \leq f_1 \leq 1;\ 0 \leq f_2 \leq 1;\ k = 0 \text{ or } 1.$$

It is obvious that the equalities (7) provide a zero value of k in case $f_1 < f_2$ and then it follows from (13) that $f_3 = 1$. At $f_1 > f_2$ the same inequalities lead to $k = 1$, and (13)– to $f_3 = 1 - f_1 + f_2$. Hence (13), (14) and (15) interpret unambiguously (12).

2.2. Many-valued logical system of L. Brower and A. Heyting

An initial point for the construction of this system is the concept of L. Brower that the unlimited action of the law for excluded of the third is in power only for this part of mathematics, which is a limited mathematical system. This fact reflects directly on the way negation and implication are formed in many-valued logical systems.

The disjunction and conjunction in Brower-Heyting's system are the same as in Lucasiewicz' system, i.e., they can be interpreted by relations from (1) up to (10).

The negation $\neg f_1$ in the system discussed can be defined as:

(16) $$\neg f_1 = f_3 = \begin{cases} 0, \text{ if } f_1 = 1; \\ 1, \text{ if } f_1 = 0; \\ 0, \text{ if } 0 < f_1 < 1. \end{cases} \qquad \begin{cases} 0, \text{ if } f_1 > 0; \\ 1, \text{ if } f_1 = 0. \end{cases}$$

This negation can be represented by the relations:

(17) $$f_3 = k\,(1 - f_1);.$$

(18) $$0 < k + f_1 \leq 1$$

under conditions (15).

In case $f_1 > 0$ it follows from (15) and (18) that $k = 0$, which leads in (17) to $f_3 = 0$. Otherwise ($f_1 = 0$), it follows from the same relations that $k = 1$ and $f_3 = 0$.

The implication in Brower-Heyting's system is defined relatively easy by

(19) $$f_1 \to f_2 = f_3 = \begin{cases} 1, \text{ if } f_1 \leq f_2; \\ f_2, \text{ if } f_1 > f_2. \end{cases}$$

The representation of $f_1 \to f_2$ by equalities and inequalities is done as

(20) $$f_3 = 1 + k\,(f_2 - f_1)$$

satisfying inequalities (14) and (15).

The justification for the correspondence of the implication between (19) and (20) can be done by considerations, similar to those justifying the correspondence between (12) and (13).

3. Many-valued logical system of E. Post

When constructing his many-valued logical system, E. Post has used formal considerations according to which the arguments obtain values from the first n numbers of the natural series $N_n = \{1, 2, ..., n\}$. The functions of these arguments obtain values from the same set N_n.

At $n=2$ the many-valued logic of Post generalizes the classic two-valued logic. In Post's system the truth is represented by 1, and the false - by the number n. This produces formally a reverse way of defining the disjunction and conjunction in comparison with Lucasiewicz' system. The disjunction in Post's system is defined by relations (3), (6), (10) and the inequalities:

(21) $$0 \le f_3 \le f_1 \le n;\ 0 \le f_3 \le f_2 \le n.$$

where

(22) $$f_1 \in N_n; f_2 \in N_n; f_3 \in N_n .$$

The conjunction is interpreted by relations (3), (6), (7), (8), (22) and the inequalities:

(23) $$0 \le f_1 \le f_3 \le n;\ 0 \le f_2 \le f_3 \le n.$$

There are two types of negation in Post's system. For the first of them

(24) $$\neg f_1 = f_3 = \begin{cases} f_1 + 1, \text{ if } f_1 < n; \\ 1, \text{ if } f_1 = n. \end{cases}$$

It can be represented by requirements (6), (22) and

(25) $$f_3 = 1 + k f_1; \quad f_1 < k + f_1 \le n .$$

The zero value of k is obtained at $\neg f_1 = f_1 = +1$; $f_1 < n$, and k is equal to 1 in the case when $\neg f_1 = 1$; $f_1 = n$.

The second type of negation is defined by

(26) $$\neg f_1 = f_3 = n - f_1 + 1 \text{ for every } f_1 \in N_n .$$

In Post's system the implication is defined as in Lucasiewicz' one, accounting the specifics in the interpretation of the truth and false respectively by 1 and n. The implication has the following form in the formalism proposed in the current paper:

(27) $$f_1 \to f_2 = f_3 = \begin{cases} 1, \text{ if } f_2 \le f_1; \\ (1 - f_1 + f_2), \text{ if } f_2 > f_1. \end{cases}$$

It can be obtained by requirements (6), (22) and

(28) $$f_3 = 1 - k\,(f_1 - f_2);$$

(29) $$0 \le k + (1/n)\,(f_1 - f_2) < 1.$$

The analysis of these relations indicates that $k=0$ corresponds to the case $f_2 < f_1$, when $f_3 = 1$, and k is equal to 1 when $f_2 > f_1$ and $f_3 = 1 - f_1 + f_2$.

In a similar way, using the flow functions $\{f_i\}$ from the graph in Fig. 1, some logical operations in other many-valued logical systems can also be formalized.

4. Network-flow interpretation of many-valued logical formulas

The formalism for representation of many-valued logic decribed enables the solution of the following two problems:

a) with the help of formulas with apriori known true values to define the true value of a logical expression which contains these formulas;

b) given an apriori set true value of the logical expression to define the true value of one of the formulas included in it.

For this purpose the initial graph should be constructed so that the formulas in brackets precede the more common expressions.

Let the following expression in many-valued logic of Lucasiewicz be given as an example:

(30) $$E = ((A \vee B) \wedge C) \rightarrow D,$$

where A, B, C, D and E are formulas, connected with the logical operations disjunction, conjunction and implication.

Fig. 2 shows the graph with 8 nodes and 7 arcs corresponding to (30), on which the logical operations that are interpreted at the respective node, are conditionally shown.

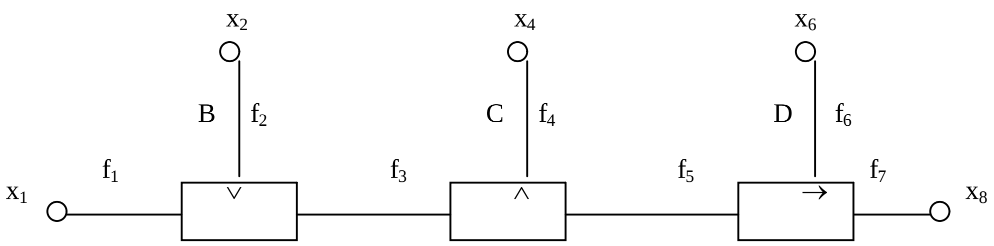

Fig. 2

The conservation equations (1) of Lucasiewicz' logic for this graph have the following form:

(31) $$f_3 = k_1 (f_1 - f_2) + f_2; \quad f_5 = k_2 (f_3 - f_4) + f_5;$$

(32) $$f_3 = 1 - k_3 (f_5 - f_6).$$

They follow from (3) and (13).

In order to define the network flow, it is necessary to satisfy the following constraints:

(33) $$0 \le f_1 \le f_3 \le 1; \; 0 \le f_2 \le f_3 \le 1;$$

(34) $$0 \le f_5 \le f_4 \le 1; \; 0 \le f_5 \le f_3 \le 1;$$

(35) $$0 \le k_3 + f_6 - f_5 \le 1; \; 0 \le f_4 \le 1.$$

(36) $$0 \le f_5 \le 1; \; 0 \le f_6 \le 1; \; k_1, k_2, k_3 = 0 \text{ or } 1.$$

If it is assumed that

$$A = f_1 = 0{,}9; \; B = f_2 = 0{,}7; \; C = f_3 = 0{,}6; \; D = f_6 = 0{,}4.$$

Then, solving the extremum problem $f_7 \rightarrow \max$, under constraints (31)-(36), $f_7 = 0{,}8$ will be obtained.

The illustrated network-flow interpretation of the implication shows that in an analogous way the rules Modus Ponens and Modus Tallens can be represented for many-valued logical inference.

4. Conclusion

The present paper discusses the possibilities for application of some quantitative methods in the definition of the true value of many-valued logical formulas. The use of a generalized network-flow model is proposed for interpretation of disjunction, conjunction, negation and implication in the logical systems of J. Lucasiewicz, L. Brauer-A. Heyting and E. Post. This enables the following:

a) with the help of formulas with apriori known true values to define the true value of a common formula, which contains these formulas;

b) given apriori set true values of the common and some other formulas to define the true value of one formula included in it.

It is shown that the generalized network-flow model suggested can be used to interpret a many-valued logical inference as well.

References

1. Blair, C. E., R. G. Jeroslaw. Some results and experiments in programming techniques for propositional logic. – Comp. and Oper. Res., **13**, 1986, No 5, 633-645.
2. Jeroslaw, R. G. Logic-Based Decision Support. North-Holland, N.Y., 1989.
3. Hooker, L. N. A Quantitative approach to logical inference. – Decision Support System, **4**, 1988, 45-69.
4. Sgurev, V. S. Network flow approach to clause inference. – Probl. of Eng. Cybernetics and Robotics, 1997, No 5, 3-10.
5. Jensen, P. A., J. W. Barnes. Network Flow Programming. J. N. Y., Wand Sons, 1980.
6. Rescher, N. Many-valued logic. N. Y., Mc Grow Hill, 1969.
7. Post, E. L. Introduction to a general theory of elementary propositions. Amer. – Journal of Mathematics, **XLIII**, 1921, No 3.
8. Cleene, S. C. Mathematical Logic. N.Y., John Wiley & Sons, 1967.
9. Zinoviev, A. A. Philosophic Problems of Many-valued Logic. Academy of Sciences of USSR, 1960 (in Russian).

Метод колличественной интерпретации многозначных логических систем

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(Р е з ю м е)

В работе предлагается метод колличественной потоковой интерпретации многозначных логических систем. С помощью обобщенной потоковой модели с выбранными соответствующим образом равенствами и неравенствами представлены логические операции дизьюнкция, коньюнкция, импликация и отрицание в многозначных логиках Лукасевича, Брауэра-Гейтинга и Поста. Определение многозначных логических формул сведено к экстремальной потоковой задаче на сети.

Даны возможности применения колличественного потокового метода для осуществления многозначного логического вывода.