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Obtaining Weak Pareto Points for Multiobjective Linear Fractional Programming Problems*

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1. Introduction

Linear fractional criteria are frequently encountered in finance, marine transportation, water resources management, health care, etc. [13]. The real decision making in these fields must take into account linear fractional (ratio) criteria very often.

The linear fractional programming (LFP) problem is defined as follows:

(1)
$$\max \left| f(x) = \frac{p(x)}{q(x)} \right|$$

s.t.
$$x \in S \subset R^n,$$

where $p(x)$ and $q(x)$ are linear functions and the set S is defined in the following way:

$$S = \{ x \mid Ax = b, \ x \geq 0 \}.$$

Here A is a real valued $m\times n$ matrix, $b \in R^m$. We suppose that S is a nonempty bounded polyhedron. The maximal value of $f(x)$ on S is denoted by $f_{\max}$.

Many authors have proposed algorithms for solving problem (1), for example: [5, 11, 17] and others. Comparative investigations of such algorithms can be found in [1, 3]. Additional information concerning especially the "bad points" is given in the paper of [14]. A point $x^1 \in S$ is called a "bad point" if $f(x) \to \infty$ when $x \to x^1$. A complete simplex type algorithm for solving problem (1) is presented in [1].

B a z a r a a and S h e t t y [2] have shown that the goal function in (1) has several important properties – it is (simultaneously): pseudoconvex, pseudoconcave, quasiconvex, quasiconcave, strict quasiconvex and strict quasiconcave. This means that the point, that satisfies the Kuhn-Tucker conditions for the maximization problem gives the global maximum on the feasible set. In addition, each local maximum is in the same time a global

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one on the feasible set. This maximum is obtained at an extreme point of S.

The multiobjective linear fractional programming (MOLFP) problem can be written as follows:

$$\max \frac{p_1(x)}{p_2(x)},$$

$$\max \frac{q_1(x)}{q_2(x)},$$

$$\vdots$$

$$\max \frac{p_h(x)}{q_k(x)}$$

s.t.

$$x \in S.$$

Here $S \subset R^n$ is a nonempty bounded polyhedron (as in problem (1)). All $p_i(x)$ and $q_i(x)$ are linear functions. We denote $f_j(x) = p_j(x) / q_j(x)$ $(\forall i)$ and suppose that $q_i(x) > 0$, $\forall x \in S$, $i=1, 2, ..., k$. A description of these problems, some basic information and many examples can be found in [13]. **Nykowski and Zolkiewski** [12] have proposed a replacing multiobjective linear programming problem and a compromise procedure for its solving. Several years later **Dutta, Rao** and **Tiwari** [6] have shown that computationally some of these results can be improved for the case when the denominators are identical. **Choo** has shown that the weak efficient set for problem (2) is not always a union of polyhedrons, it may contain some nonlinear parts [13]. The explicit description of the weak efficient set can be very useful but it is often a hard problem to get such description. An advantage of the weak efficient set of problem (2) is that it is always a closed set. (The efficient set may not be closed. [13]). A nonlinear programming technique and the reference point method are proposed here for obtaining weak efficient points for problem (2).

2. Analysis of the MOLFP problem using an auxiliary nonlinear programming problem

Let us consider problem (2). We can try to use the reference point method for an analysis of this problem, thus we formulate the following nonlinear programming problem

$$\min D$$

s.t.

(3)

$$D > b_i(r_i - f_i(x)), \forall i,$$

$$x \in S.$$

Here $b_i > 0$ $(\forall i)$, the numbers r_i are the reference point components, they satisfy the following inequalities:

$$r_i > \max f_i(x), \ \forall i.$$

It can be seen that the solution of problem (3) determines weak efficient points for problem (2). Really suppose that x^1 is a solution of problem (3), that gives the minimal value $D^{\min}$, but x^1 is not a weak efficient point. Then there exists another point

$x^2 \in S$, such that $f_i(x^2) > f_i(x^1)$, $(\forall i,)$. Therefore it is obviously that for the corresponding value D^2 we get $D^2 < D^{min}$, and this is a contradiction.

Definition 2. Consider the function $h: S \to E_1$, where S is a nonempty convex set in E_n. The function h is called strict quasi convex, if for each two points x^1, $x^2 \in S$, such that $h(x^1) \neq h(x^2)$, the following inequality holds:

$$h(\lambda x^1 + (1-\lambda) x^2) < \max \{h(x^1), h(x^2)\} \text{ for all } \lambda \in (0,1).$$

The functions $g_i(x) = b_i (r_i - f_i(x))$, $\forall i$, are strict quasi convex because $f_i(x)$ are linear fractional [2].

It is obvious that in problem (3) the minimum of the following function is searched

$$\varphi(x) = \max_i [b_i (r_i - f_i(x))] = \max_i [g_i(x)], \; i = 1, 2, ..., m.$$

Theorem. Let $S \subset E_n$ be a nonempty convex set. Suppose that the functions $g_i(x)$ $(\forall i, \forall x \in S)$ are strict quasi convex. Then the function $\varphi(x) = \max_i [g_i(x)]$ is strict quasi convex, too.

Proof. Let $0 < \lambda < 1$, x^1, $x^2 \in S$,. Then

$$\varphi(\lambda x^1 + (1-\lambda) x^2) = \max_i g_i(l\, x^1 + (1-l)\, x^2) <$$
$$< \max_i [\max(g_i(x^1), g_i(x^2)] = \max_i [\max g_i(x^1), \max_i g_i(x^2)] =$$
$$= \max[\varphi(x^1), \varphi(x^2)] \; \blacklozenge$$

Therefore in problem (3) we have to minimize a strict quasi convex function on the convex set S. *Each local minimum of a strict quasi convex function is in the same time a global minimum of this function on the feasible set S* [2]*.* This means that we can solve problem (3) using nonlinear programming algorithms that give local minimum. The obtained solution will give a weak Pareto point for problem (2) and a corresponding weak efficient point.

3. Numerical example

The Choo's example described in [13] will be used here for illustration purposes. This example is:

$$\max(f_1 = x_1 / x_2)$$
$$\max(f_2 = x_3)$$
$$\max(f_3 = -(x_1 + x_3) / (1 + x_2))$$

s.t.

$$1 \le x_1, x_2, x_3 \le 4.$$

The feasible set S is determined by the above given constraints. The weak efficient set E^w is described as follows [13]:

$$E^w = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5,$$

where

$$U_1 = \{ x \in S \mid x = (a, b, c), \ a = bc \},$$
$$U_2 = \{ x \in S \mid x = (4, b, c), \ bc \geq 4 \},$$

U_3 is the convex hull of the points (1, 4, 4), (1, 4, 1), (4, 4, 1), (4, 4, 4),

U_4 is the convex hull of the points (4, 1, 4), (1, 1, 4), (1, 4, 4), (4, 4, 4),

U_5 is the convex hull of the points (4, 1, 1), (4, 1, 4), (1, 1, 1).

In order to get weak efficient points for this problem we use formulation (3). The computations were made by program NELI. The feasible set is given as shown above. The functions f_i are written in general mode in the constraints containing the variable D. These functions are determined explicitly by separately written constraints. On the other hand the reference point components are numerically written in the constraints containing the variable D. In addition $b_i \equiv 1$ for all i. Table A contains data illustrating the behaviour of the solution.

Table A

1	r_1	r_2	r_3	x_1	x_2	x_3	f_1	f_2	f_3
2	1	2	3	4	5	6	7	8	9
3	5	5	1	1,999922	1,00	1,999959	1,999959	1,99959	−2,000041
4	6	5	1	2,75	1,00	1,75	2,75	1,75	−2,25
5	5	6	1	4,00	2,171158	2,842325	1,842325	2,842325	2,157675
6	5	5	2	1,499922	1,00	1,499959	1,499922	1,499959	−1,500041

The components of four reference points: (5, 5, 1), (6, 5, 1), (5, 6, 1), (5, 5, 2) are written in columns 1, 2, 3 and in rows number 3, 4, 5, 6. The same rows and columns 4, 5, 6 contain the corresponding feasible points determined by the solution of problem (3) obtained with the corresponding reference point. The last three columns contain the corresponding criteria values. The comparison with the given explicit description of the weak efficient set shows that all feasible points written in Table A are weak efficient. It must be pointed out that all used reference points dominate the ideal point for the problem.

Table A very clearly illustrates the effects of increasing of one reference point component keeping the rest unchanged. Row 4 contains a reference point with first component increased with respect to the reference point in row 3. This leads to increasing the value of the first criterion (rows 3 and 4, column 7). The same effect can be seen for the second and the third reference point component. This effect is generally described in the paper [8].

It must be added here that the computations made with the nonlinear programming formulation (3) were compared and confirmed by computations based on the usage of linear programming technique.

4. Some comments and conclusion

The paper [8] contains a result concerning the usage of reference points for the analysis of nonlinear multiobjective optimization problems. It is shown in the paper that the obtained value of a given criterion can be improved by a correspondingly chosen reference point. This result is valid for the considered here multiobjective problems and is illustrated by the example. The same result gives a way to move in the weak Pareto set.

The paper [8] contains, in addition, a result about the Pareto points attainability. This result is valid for the problems considered here, too. The solution of problem (3) determines weak efficient points for problem (2) (and weak Pareto points, of course). In general, if the reference point is close to a Pareto point, then the solution determines a Pareto point. Thus from practical point of view it can be said that weak Pareto points are attainable.

It is worth noting that problem (3) must be fully solved – in the following sense. It is not sufficient to find (or to estimate) the needed minimum only, without determining the corresponding argument. This argument determines the needed weak Pareto or weak efficient point.

The proposed way for obtaining weak Pareto (weak efficient) points seems to be more attractive in the cases, when the weak efficient set has a large number of extreme points and it is a hard problem to get a full description of this set.

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Нахождение слабых точек Парето для задач многокритериального дробно-линейного программирования

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(Р е з ю м е)

Для анализа задачи многокритериального дробно-линейного программирования (все критерии максимизируются) предлагается использовать известную скалярную оптимизационную задачу, решение которой определяет слабые Паретовские точки (а также и слабоэффективные точки). В рассматриваемом случае в этой скалярной задаче минимизируется строго квазивыпуклая функция, что позволяет использовать алгоритмы нелинейного программирования, дающие локальный минимум.