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An Interactive Reference Direction Algorithm of Nonlinear Integer Multiobjective Programming*

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1. Introduction

The basic algorithms having established themselves as fundamental in solving integer multiple objective programming problems are interactive (refer to [1, 4, 5, 6, 9]. Each interactive algorithm performs a sequence of iterations consisting of two phases: a computational phase and a dialogue phase. During the computational phase one or several solutions are generated, usually solving a scalarizing problem. They are submitted for evaluation to a person, called decision maker (DM) during the dialogue phase. If the DM accepts any solution as the most preferred one, the iteration sequence terminates. In the opposite case the DM has to present quantitative or qualitative information concerning his/her preferences. This information is necessary for the formulation of a new scalarizing problem which is solved at the next iteration and the solution obtained is presented to the DM for evaluation. The interactive algorithm terminates when the most preferred solution is found.

The nonlinear problems with continuous variables and the convex integer (linear and nonlinear) problems are NP-hard problems (see [2, 3]). The exact algorithms for their solving have exponential computational complexity. The integer problems are characterized by the fact that the finding of a feasible solution is so difficult as the finding of an optimal solution.

The development of interactive algorithms for solving multiple objective nonlinear and integer (linear and nonlinear) programming problems requires bindingly taking into account the time necessary for solving the scalarizing problems. The dialogue with the DM, even very convenient, may not be held. This may happen in case the DM cannot wait very long when solving a scalarizing problem.

One approach for overcoming the difficulties concerning the computational complexity of the single objective linear integer problems is proposed in [5, 7]. It may be the most innovative among the interactive algorithms known at the present time, which are designed to solve multiple objective linear integer programming problems (see for example [4, 6, 8, 9]. The main feature of this approach is that single objective linear

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problems with continuous variables are solved during the training (learning) process and the solutions obtained are submitted to the DM for evaluation. These problems are easy solvable. They are used under the assumption that the objective function values for the scalarizing problem solutions with continuous variables differ comparatively little from the solutions with integer variables. It is supposed also that the DM prefers to deal with the objective functions but not with the variables. The advantage of these interactive algorithms is that the quality of the dialogue with the DM does not get worse with respect to the type of information required from and produced by the DM and that it is improved with respect to the computational time expended to obtain a new solution for evaluation.

Unfortunately this approach loses its advantages when multiple objective convex nonlinear integer problems must be solved. The scalarizing convex nonlinear integer problems are difficult solvable. The corresponding single objective convex nonlinear problems with continuous values are difficult solvable too. Therefore it is unattractive to use such problems for DM's training.

The paper suggests an interactive algorithm for solving multiple objective convex nonlinear integer problems, in which the difficulties concerning their computational complexity are overcome to some extent. It can be referred to the reference direction algorithms. The DM sets his preferences as aspiration levels of the separate criteria. The reference direction is defined by the aspiration point in the criteria space and the solution found at the previous iteration. On the basis of the reference point a scalarizing problem is constructed with two specific properties. The first one is that feasible integer solutions of this problem lie close to the efficient frontier, and the second one is that the solution found at the previous iteration is its feasible solution. The first property of the problem enables the application of an approximate algorithm of polynomial complexity. The approximate solutions found with the help of this algorithm can be used for evaluation by the DM, especially in the learning process. These are the so called "near (weak) efficient" solutions. The second property enables the application of an approximate algorithm of "tabu search" type (refer to [10]), which is quite efficient at given initial feasible solution. An exact algorithm can be used for the scalarizing problem solving at the last iteration or at some intermediate iterations in case the DM wishes.

2. The problem discription

The algorithm proposed in this paper is designed to solve multiple objective convex nonlinear integer programming problems, which may be stated in the form:

(1) $$\max \{ f_k(x) \mid k \in K \}$$

subject to:

(2) $$g_i(x) \le 0,\ i \in M,$$

(3) $$0 \le x_j \le d_j,\ j \in J,$$

(4) $$x_j \text{ – integer},\ j \in J,$$

where $f_k(x)$, $k \in K$, are concave functions; $g_i(x)$, $i \in M$, are convex functions; $K=\{1, 2, ..., p\}$, $M=\{1, 2, ..., m\}$, and $J=\{1, 2, ..., n\}$. The symbol "max" means that each objective function has to be maximized. The constraints (2)–(4) define the feasible set X.

A few definitions are given primarily to improve the clarity of the text:

Definition 1. The solution x is called efficient if there does not exist another solution $\bar{x} \neq x$, such that the inequalities

$$f_i(\bar{x}) \ge f_i(x) \text{ for each } i \in K,$$

$$f_i(\bar{x}) > f_i(x) \text{ for at least one } i \in K$$

hold.

Definition 2. The solution x is called weak efficient if and only if there does not exist another solution $\bar{x} \neq x$, such that

$$f_i(\bar{x}) > f_i(x) \quad \text{for each } i \in K.$$

Definition 3. The p-dimensional vector $f(x)$ with components $f_i(x)$, $i \in K$; is called (weak) nondominated, if x is an (weak) efficient solution.

Definition 4. The preferred (weak) efficient solution is the (weak) efficient solution chosen by the decision maker (DM) at the current iteration of the proposed algorithm.

Definition 5. The most preferred (weak) efficient solution is the (weak) efficient solution that satisfies the preferences of the DM to the greatest degree.

Definition 6. The reference direction is defined by the difference between the reference point (aspiration levels) given by the DM and the (weak) nondominated solution obtained at the previous iteration.

The problem (1)–(4) does not possess an analytically well defined optimal solution. Therefore it is necessary to select a solution from the set of efficient solutions. This process is subjective and depends on the DM.

Let X denote the set of solutions, which satisfy constraints (2)–(4). Let f_k denote the value of the solution found at the last iteration for the k-th, $k \in K$, objective function and $\bar{f}_k$, $k \in K$, denote its desired value (aspiration level) defined by the DM at the current iteration. Further, let

$$K_1 = \{ k \in K \mid \bar{f}_k > f_k \},\ K_2 = \{ k \in K \mid \bar{f}_k < f_k \},$$

$$K_3 = \{ k \in K \mid \bar{f}_k \geq f_k \},\ K = K_1 \cup K_2 \cup K_3 .$$

The set K_1 contains the indices of the criteria, the values of which the DM wishes to improve, while the set K_2 contains the indices of the criteria on the account of which this improvement can be done. The set K_3 contains the indices of the criteria which the DM is not inclined to deteriorate.

The following single objective problem is proposed to obtain a (weak) efficient solution:

Minimize

(5) $$S(x) = \max[\max_{k \in K_1} (\bar{f}_k - f_k(x))/(\bar{f}_k - f_k), \max_{k \in K_2} (f_k - f_k(x))/(f_k - \bar{f}_k)]$$

subject to:

(6) $$f_k(x) \geq \bar{f}_k,\ k \in K_3 .$$

(7) $$x \in X .$$

It should be noted that problem (5)–(7) has a feasible solution if the feasible set X is non-empty, and has an optimal solution if the objective functions $f_k(x)$, $k \in K$, are finite over X.

The basic feature of the scalarizing problem (5)–(7) is the minimization of the maximal standardized deviation of the searched solution $f_k(x)$, $k \in K$ and the modified aspiration point in the objective space. The modified aspiration point differs from the DM aspiration point in this, that the aspiration levels $\bar{f}_k, k \in K_3$ are replaced by the values of the criteria $f_k, k \in K_2$, they possess in the last solution found. In other words, though the DM accepts the deterioration of the criteria $f_k(x)$, $k \in K$ upto $\bar{f}_k$, the last levels of these criteria are included in the modified aspiration point. Thus with the scalarizing problem (5)–(7) the DM strategy "little benefit – no great loss" is realized. This strategy of DM behaviour enables the overcoming of some computing problems, connected with the solution of multiobjective convex nonlinear integer problems. In this way the dialogue with the DM is also improved.

Theorem. The optimal solution of (5)–(7) is a weak efficient solution for (1)–(4)

Proof. If $K_3 \neq \varnothing$ the proof is obvious. Let $K_3 = \varnothing$. Let x^* be the optimal solution of problem (5)–(7). Then the following inequality holds:

$$S(x^*) \leq S(x) \text{ for each } x \in X. \tag{8}$$

Let us assume that x^* is not a weak efficient solution for problem (1)–(4). There exists a point $x' \in X$, such that

$$f_k(x^*) < f_k(x') \text{ for } k \in K. \tag{9}$$

The following relation

$$S(x') = \max[\max_{k\in K_1}(\bar{f}_k - f_k(x'))/(\bar{f}_k - f_k), \max_{k\in K_2}(f_k - f_k(x'))/(f_k - \bar{f}_k)]$$

$$= \max[\max_{k\in K_1}((\bar{f}_k - f_k(x^*))+(f_k(x^*) - f_k(x')))/(\bar{f}_k - f_k),$$

$$\max_{k\in K_2}((f_k - f_k(x^*))+(f_k(x^*) - f_k(x')))/(f_k - \bar{f}_k)] \tag{10}$$

$$< \max[\max_{k\in K_1}(\bar{f}_k - f_k(x^*))/(\bar{f}_k - f_k), \max_{k\in K_2}(f_k - f_k(x^*))/(f_k - \bar{f}_k)] = S(x^*)$$

is obtained after transforming the objective function $S(x)$ of problem (5) – (7), using inequality (9).

It follows from (10) that $S(x') < S(x^*)$, which contradicts (8). Hence x^* is a weak efficient solution for problem (1) – (4).

The problem (5) – (7) can be stated as the following equivalent mixed integer convex nonlinear programming problems:

$$\min \alpha \tag{11}$$

$$(\bar{f}_k - f_k(x))/(\bar{f}_k - f_k) \leq \alpha,\ k \in K_1, \tag{12}$$

$$(f_k - f_k(x))/(f_k - \bar{f}_k) \leq \alpha,\ k \in K_2, \tag{13}$$

$$f_k(x) \geq f_k,\ k \in K_3, \tag{14}$$

$$x \in X, \tag{15}$$

$$\alpha \text{ – arbitrary.} \tag{16}$$

When the problem (5)–(7) has no solution, then the problem (11)–(16) also has no solution. This is due to the fact that both problems have the same original constraints. When the problem (5) –(7) has a solution, then (11) – (16) has a solution and the optimal values of their objective functions are equal. The last statement is derived from the following lemma:

Lemma. The optimal values of the objective function of problems (5) – (7) and (11)–(16) are equal.

$$\min_{x\in X} \alpha = \min_{x\in X}\{\max[\max_{k\in K_1}(\bar{f}_k - f_k(x))/(\bar{f}_k - f_k), \max_{k\in K_2}(f_k - f_k(x))/(f_k - \bar{f}_k)]\}$$

Proof. From (12) follows that

$$\alpha \geq (\bar{f}_k - f_k(x))/(\bar{f}_k - f_k),\ k \in K_1.$$

Since this inequality is valid for each $k \in K_1$, it follows that

$$\alpha \geq \max_{k \in K_1} (\bar{f}_k - f_k(x))/(\bar{f}_k - f_k) \tag{17}$$

From (13) follows that

$$\alpha \geq (f_k - f_k(x))/(f_k - \bar{f}_k),\ k \in K_2.$$

Since this inequality is valid for each $k \in K_2$, it follows that

$$\alpha \geq \max_{k \in K_2} (f_k - f_k(x))/(f_k - \bar{f}_k). \tag{18}$$

From (17) and (18) it follows that

$$\alpha \geq \max[\max_{k \in K_1} (\bar{f}_k - f_k(x))/(\bar{f}_k - f_k), \max_{k \in K_2} (f_k - f_k(x))/(f_k - \bar{f}_k)].$$

If x^* is an optimal solution for (11) – (16), then

$$\min_{x \in X} \alpha = \max[\max_{k \in K_1} (\bar{f}_k - f_k(x^*))/(\bar{f}_k - f_k), \max_{k \in K_2} (f_k - f_k(x^*))/(f_k - \bar{f}_k)]$$

because otherwise α can be decreased further.

Furthermore, the right-hand side of the proceeding equality is equal to

$$\min_{x \in X} \max[\max_{k \in K_1} (\bar{f}_k - f_k(x))/(\bar{f}_k - f_k), \max_{k \in K_2} (f_k - f_k(x))/(f_k - \bar{f}_k)]$$

which proves the lemma.

The single-objective problem (11)–(16) has two positive properties in a computing aspect. The first one is that the solution obtained in the previous iteration is the feasible solution for the single-objective problem being solved at the current iteration. This facilitates the functioning of the single-objective integer algorithms, since they can always start with a feasible solution. This is particularly important for the approximate algorithms of "tabu search" type.

The second property is connected with the fact that the feasible solutions of problem (11)–(16) are near to the efficient frontier of the multiobjective problem (1)–(4). This property enables the use of approximate single-objective integer algorithms since the approximate solutions of problem (11)–(16) found with their help are localized near to the efficient surface of problem (1)–(4). The application of approximate integer algorithms of tabu search type for the solving of problem (11)–(16) in some cases will lead to the obtaining of near (weak) nondominated solutions and at the same time it will decrease considerably the waiting time in the dialogue with the DM. This is especially appropriate in the initial iterations, when the DM is learning. During the learning period the interruption of the approximate single-objective algorithm operation is possible and the use of the approximate solutions obtained upto this moment.

3. The algorithm proposed

The reference direction approach used helps the DM to make decisions directing him/her quickly to the most preferred solution. The number of iterations and of the single objective problems solved is decreased in this way. In addition the algorithm uses an approximate procedure for solving single objective integer problems which speeds up its performance. At each iteration the DM gives a reference point. The original multiobjective problem is reduced to a series of scalarizing problems which are single objective convex integer problems. Each solution is evaluated and the DM decides if he/she wants to change the reference direction or to stop.

If the current nondominated solution seems to the DM far from the most preferred solution, a new single objective convex nonlinear integer problem is solved by an approximate polynomial algorithm. The solutions obtained are near (weak) nondominated solutions. The quality of these solutions depends on the algorithm used to solve the single objective problems. When the DM feels that a given near (weak) nondominated solution is close to the most preferred solution, he/she may use an exact algorithm to obtain the optimal solution of the current single objective problem. The last near (weak) nondominated solution found is used as a starting point in the exact algorithm. The search procedure goes on until the most preferred solution is found. To solve a "large" multiple objective problem, the DM may only use an approximate algorithm for solving the single objective convex integer problems.

The steps of the algorithm proposed can be stated as follows:

Step 1. If an initial integer feasible solution is available for the problem (1–4) go to Step 2. Otherwise set $f_k = 0$, and $\bar{f}_k = 1$, $k \in K$. Solve the problem (5–7) using the approximate algorithm to obtain a feasible initial solution. If the DM is satisfied with this solution, go to Step 7; otherwise go to Step 2.

Step 2. Ask the DM to provide the new aspiration levels.

Step 3. Ask the DM to choose the type of the algorithm – exact or approximate. If the DM has selected the exact algorithm, go to Step 5.

Step 4. Ask the DM to specify t – the maximal number of near (weak) nondominated solutions the DM wants to see along the reference direction at the current iteration. Go to Step 6.

Step 5. Solve the problem (5)–(7). Show the (weak) nondominated or near (weak) nondominated solution obtained (in case the computing process has been interrupted) to the DM. If he/she approves this solution, Stop. Otherwise go to Step 2.

Step 6. Solve the problem (5)–(7). Submit to the DM t in number (if more than t are obtained) near (weak) nondominated solutions. If the DM is satisfied with one of them, Stop. Otherwise go to Step 2.

Step 7. Stop.

Remark 1. When the DM sets the aspiration levels in Step 2, the separating of the criteria into three groups is required, depending on his attitude to the values of these criteria, i.e., which criteria he wants improved, which may be deteriorated and which cannot be weakened.

Remark 2. In Step 3 it is necessary the DM to be aware of the fact that the choice of an exact algorithm leads to the obtaining a nondominated integer solution for evaluation by him, but on the account of longer time expended for its obtaining. On the contrary, when an approximate algorithm is chosen, the time for the solution getting is considerably smaller, but on the account of possible deterioration of the quality of this solution.

Remark 3. If an approximate algorithm is chosen, in Step 4 it is defined how many near (weak) nondominated solutions for evaluation along the reference direction the DM wants to gain. This possibility is included due to two reasons. The first one is that all the solutions obtained are approximate solutions. The second one is that during the learning process it is of particular use for the DM to consider and evaluate more solutions.

Remark 4. When using an exact algorithm in Step 5, problem (5)–(7) is solved in order to obtain a (weak) nondominated solution. If the DM decides that the solution time is too long, he may interrupt the computing process and evaluate the last approximate solution obtained.

The proposed algorithm for solving multiobjective convex nonlinear integer problems has a lot of advantages. The number of the single objective problems solved is equal to the number of the aspiration points. The DM must define only which criteria and by what amount to be deteriorated and which must not be weakened. The DM operates in the criteria space, which in most of the cases is convenient for him, since in general the criteria have physical or economic aspect. The application of approximate algorithms for single objective problems solving facilitates the dialogue of the DM, which influences positively his behaviour in the process of multiple objective problem solving. The presenting at each iteration of several, though near (weak) nondominated solutions along the reference direction enables the faster teaching of the DM with respect to the set of near (weak) nondominated set and hence also to the set of nondominated solutions. The last confirmation is fulfilled to a great extent, because the scalarizing problems (5)–(7) being solved have quite limited feasible areas, including parts of the (weak) nondominated set. Besides this the use of an approximate algorithm of "tabu search" type, operating well in narrow feasible areas at known initial feasible solution aids the finding rather good and in many cases optimal solutions of these scalarizing problems.

One disadvantage of the algorithm proposed is maybe this one, that the DM has to take into mind the last evaluated but not approved solution when setting new aspiration levels. In other words, if he wants to obtain new solutions, the DM has to allow the deteriorating of at least one criterion in the last solution considered. Another shortcoming of the interactive algorithm proposed is the requirement to use approximate single objective algorithm, that is performing well in narrow feasible areas.

4. Concluding remarks

The interactive algorithm proposed is designed to solve convex nonlinear problems of multiple objective integer programming. It belongs to the interactive reference direction algorithms. The DM evaluates approximate solutions close to the nondominated surface in the learning process. These solutions are found by means of the used approximate tabu search algorithm. There is also a possibility to find for evaluation at some iterations nondominated solutions by means of an exact algorithm.

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Интерактивный алгоритм отправных направлений нелинейного целочисленного мультикритериального программирования

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(Р е з ю м е)

Предложен интерактивный алгоритм отправных направлений для решения выпуклых проблем многокритериального математического программирования. На каждой итерации алгоритма находятся одно или несколько приближенных целочисленных решений, вблизи от недоминированной поверхности. В случае, если лицо, принимающее решение (ЛПР), желает, он может при несколько итерациях найти и только недоминированные решения.
В алгоритме ЛПР работает только с аспирационными уровнями. Использование алгоритмов типа "Табу искания" предлагается для приближенного решения полученных скаляризирующих задач.