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Properties of the Effective Solutions of the Multicriteria Network Flow Problem

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1. Introduction

Let the network $G=\{N,U\}$ consists of a set N of n nodes and a finite set U of m directed arcs be given. Each arc is defined as an ordered pair (i, j), where i denotes the initial node and j denotes the ending node; k "cost" parameters a_{ij}^k, $k\in I_k$, are associated with each arc (i, j). The multicriteria flow problem (MFP) may be stated as follows:

$$\mathrm{MF}(X):\quad \min{}^*\,(g_1(X), g_2(X), \ldots, g_k(X))$$

subject to

$$\sum_{j\in N} x_{ij} - \sum_{j\in N} x_{ji} = \begin{cases} v & \text{if } i=s, \\ 0 & \text{if } i\neq s, t, \\ -v & \text{if } i=t, \end{cases} \tag{1}$$

$$0\leq x_{ij}\leq c_{ij},\quad (i, j)\in U, \tag{2}$$

where s is the source node and t is the terminal node (the sink),

$$g_i(X) = \sum_{(i,j)\in U} a_{ij}^{\,i} x_{ij},$$

$$v\leq v^*,$$

and v^* is the value of of the maximal flow.

Each x satisfies (1) and (2) is called a feasible solution (f.s.) of the stated problem. The solution of the problem MFP consists in the determining of all efficient solutions (flows) with a fixed value $v\leq v^*$. A flow $X=\{x_{ij}, (i, j)\in U)\}$ is an efficient solution (e.s.) or flow, if there exists no other flow X_1, $X_1\neq X$, which can improve the value of one of the functions g_i, without worsening the value of the other.

A lot of applications, which need to use a flow model could be narrowed in a natural way. Those applications could be potential application of BCFP – for example – transportation networks, where each link is associated with a cost parameter, lenght and time parameter. It is directly related to the existing traffic along the link.

When k=1 the problem is reduced to the single criterion problem for min-cost flow (MCF). In general this is a linear programming problem. Polynomial algorithms exist for their solving. The efficiency, the efficient data support of those algorithms are due to the unimodularity of the constraints matrix. That is why the class of the flow problems is distinguished in the class of the linear programming problems.

The problem MFP is a multicriteria linear programming problem. The methods for solving this class of problems can be applied to it. The solution of MFP with these methods is related with the use of new general linear constraints. They are joined with the set of flow constraints. From this joint the constraint matrix loses its specific unimodular properties. Some efficient algorithms have been designed for solving problems for a network flow with additional linear constraints. They are adaptations of the simplex method [1, 2]. The problem here is that some of the main advantages of the flow method, such as polynomiality, integer-valued solution, effective structure of the data disappear. All this leads to a subject, which is a topic for a explorer's interest [6]. The subject is – to find the efficient solutions (e.s.) for this class of problems, creating methods which have to conserve the flow structure of the constraint matrix in the solving of scalarized problems. In the present paper we invetigate the structure of the set of e.s. of the problem MF(X).

We propose an algorithm for finding the e.s. of the bicriteria network flow (BFP). There isn't large variety of methods for solving BFP. The methods, which are suggested in [3, 4] use parameterization and move to neighbouring bases. The complexity of these methods depends on the number of the e.s. Approximate methods are described in [3]. They are based on the "sandwich" algorithm, i.e. on the approximation of the set of e.s. below and up, with pseudopolynomial complexity.

2. Theoretical properties of the MNFP

Let X is a feasible flow which satisfies the conditions (1) and (2). Let us define the network G_x={N,U_x}, called residual. The arc $(i, j) \in U_x$, if in the original network G the following inequalities are satisfied: $x_{ij} < c_{ij}$ or $x_{ji} > 0$. The capacity of the arc $(i, j) \in U_x$ is $c_{ij} - x_{ij}$ or x_{ji} respectively. Every flow $Y=\{y_{ij}, (i, j) \in U_x\}$ in the network G_x satisfies the conditions

(3) $$\sum_{j\in N} y_{ij} - \sum_{j\in N} y_{ji} = 0, \text{ if } i \neq s, t,$$

(4) $$-x_{ij} \leq y_{ij} \leq c_{ij} - x_{ij}, \ (i, j) \in U.$$

We define the "cost" $b_{ij}^{\,p}$ of the arc $(i, j) \in U_x$:

$$b_{ij}^{\,p} = a_{ij}^{\,p} \text{ if } x_{ij} < c_{ij} \text{ and } b_{ij}^{\,p} = -a_{ij}^{\,p} \text{ if } x_{ji} > 0.$$

Lemma 1. If the solution X of MFP with a value v is e.s., then no cycle σ exists in the network G_x, for which

(5) $$g_i(\sigma) \leq 0, \ i \in I_k \backslash \{i_1\} \text{ and } g_{i1}(\sigma) < 0.$$

Proof. If there exists at least one cycle which satisfies (5), then the flow X+$X(\sigma)$ is of value v and the following is fulfilled:

$$g_i(X+X(\sigma)) \leq g_i(X) \text{ for } i \in I_k \backslash \{i_1\},$$

$$g_{i1}(X+X(\sigma))<0,$$

i. e. X is not an e. s. for MFP.

MFP is a linear multicriteria problem, its solution is naturally integer for integer data. We can relax the integrality conditions, The solving of the relaxed MFP problem would find all basic e. s., which are integer. The problem of finding the rest of the integer e. s. still remains. In order to describe the whole set of e. s. of the problem, theorem 2 and 3 could be applied.

Let X^1 be a feasible solution of the problem L_1:

$$\text{(6)}\qquad \begin{aligned} L_1:\quad & \min{}^*(g_i(X),\ i\in I_k) \\ \text{s.t.}\quad & f_i(X)=d_i,\ \ i\in I_m, \\ & 0\le x_i\le c_i,\ \ i\in I_n, \end{aligned}$$

where the functions $g_i(X)$, $f_i(X)$ are separable, i. e. $f_i(X+Y)=f_i(X)+f_i(Y)$.

Let X^2 be an e. s. of the problem L_2:

$$\text{(7)}\qquad \begin{aligned} L_2:\quad & \min{}^*(g_i(Z),\ i\in I_k) \\ \text{s.t.}\quad & f_i(Z)=0,\ \ i\in I_m, \\ & -x_i\le z_i\le c_i-x_i,\ \ i\in I_n. \end{aligned}$$

The following theorem is valid.

Theorem 1. X is an e.s. of the problem L_1, where $X=X^1+X^2$.

Proof. It is evident from (7) that: $0\le x_{ij}^1+x_{ij}^2\le c_i$, i. e. X is a f.s. of L_1. It is assumed, that X is not an e. s. of L_1, i. e. there exists a solution Y of L_1 such that:

$$\text{(8)}\qquad g_i(Y)\le g_i(X),\ i\in I_k\backslash\{i_1\} \text{ and } g_{i1}(Y)<g_{i1}(X) \text{ for some } i_i.$$

It is obtained from conditions (6) that:

$$-x_i^1\le y_i-x_i^1\le c_i-x_i^1,\ \ i\in I_n,$$

$Y-X^1$ satifies (7). And it is obtained from (8) that:

$$g_i(Y)\le g_i(X^1+X^2),\ i\in I_k\backslash\{i_1\} \text{ and } g_{i1}(Y)<g_{i1}(X^1+X^2) \text{ for some } i_i.$$

and

$$g_i(Y)-g_i(X^1)\le g_i(X^2),\ i\in I_k\backslash\{i_1\} \text{ and } g_{i1}(Y)-g_{i1}(X^1)\le g_{i1}(X^2) \text{ for some } i_i.$$

We conclude, that X^2 is not an e. s. of the problem L_2 which leads to a contradiction with the initial proposition.

Stating the problem L_1 for the network G, it is clear that the solution X^1 is a feasible flow with a value v in G and the problem L_2 is in fact a problem for minimal flow with a value 0 (circulation) in the residual network G_x. Or:

The sum of an feasible flow with a value v in the network G and an efficient circulation in the network G_x is an e. s. for the problem MFP.

Theorem 2. Each efficient flow in the network G can be represented as a sum of a feasible basic flow in G and an efficient circulation in the residual network G_x.

Proof. Let X be an e. s. (flow) of MFP. There exist numbers $\lambda_i\ge 0$, $i\in I_k$, $\sum_{i\in I_k}\lambda_i=1$, such that X is an optimal solution of the problem

$$\min\ F(X)=\sum_{i\in I_k}\lambda_i g_i(X)$$

s. t. (1) and (2).

For the residual network we solve the problem below:

$$\text{MF}(Y): \ \min \ F(Y)=\sum_{i\in I_k} \lambda_i \, g_i(Y)$$

s. t. (3) and (4).

Let Y is a basic e. s. for this problem. Then, there exists a spanning tree $T_{x1}=\{N,U_{x1}\}$, such that for every arc $(i, j)\in U \setminus U_{x1}$ the solution Y satisfies:

$$y_{ij} = -x_{ij} \ \text{ or } \ y_{ij} = c_{ij} - x_{ij}, \text{ i. e.}$$

(9) $$y_{ij} + x_{ij} = 0 \text{ or } y_{ij} + x_{ij} = c_{ij} .$$

It follows from (3) and (4) that $X+Y$ satisfies the conditions (1) and (2), i.e. it is a f. s. of MF(X). From (9) it follows that $X + Y$ is a basic solution of MF(X).

From the inequality

$$-X \le 0 \le C - X$$

it follows that

$$-X - Y \le -Y \le C - X - Y, \text{ i. e.}$$

the flow $-Y$ is a f. s. for the residual network G_{x+y}. Let Z be a flow in G_{x+y} for which the function $F(Y)$ has a minimal value on the set, defined by (3) and (4), i. e.

$$F(Z) \le F(-Y) \text{ or } F(Z+Y) \le 0.$$

On the other hand it is true that

$$0 \le X+(Y+Z) \le c, \text{ i. e. } X+(Y+ Z) \text{ is a f. s. for the network } G.$$

If X is a flow for which $F(X)$ takes its minimal value on the set defined by (1) and (2), then

$$F(X) \le F(X+Y+ Z) \text{ or } F(Z+Y) \ge 0.$$

Then $F(Z)= F(-Y)$, i. e. for $-Y$ the objective $F(Y)$ has minimal value on (3) and (4), or $-Y$ is an e. s. for the residual network G_{x+y}. The equality

$$X= (X+Y)+(-Y)$$

proves the theorem.

3. A method for solving BFP

The solution procedure developed here solves the problem of determining in the network all efficient integer flows from s to t with a value v. The procedure is based on the property of an efficient flow, proved in Theorem 2.

We denote by BF(X) the bicriteria flow problem. Let $X= \{x_{ij}, (i, j) \in U)\}$ be a f.s. (flow) of the problem BF(X). We will define the residual network $G_1 = \{ N_1, U_1\}$ for X and the corresponding BNFP, named BF$_x$(Y).

The proposed algorithm may be described in general as follows:

Step 1. Find an initial basic f.s. X of the problem BF(X).

Step 2. Do an existence check, using the list $E(G)$ of the already determined feasible solutions. If this solution already exists, find another. If there are no more feasible solutions, terminate. Add the solution X to the list $E(X)$.

Step 3. Define the problem BF$_x$(Y). Find all e.s. Y^i, $i\in I(i_y)$ of this problem. If there is not such a solution, go to Step 1.

Step 4. For all $i \in I(i_y)$ find the new series of e.s. X^i of BF(X) by the formula $X^i := X + Y^i$ and add them to the list $E(G)$.

Step 5. Obtain another f.s. using the solution X^{iy}.

3.1. Finding a feasible solution of the problem $F_1(X)$

To determine all basic e.s. of BF(X), we accept that it is a bicriteria linear programming problem (BLPP). The basic solutions are integer valued due to the unimodular property of the constraint matrix. We use the results described in [5]. For e.s. basic solutions X^I, i=1, 2, it is possible to rank them in increasing order of $g_1(X)$, so that:

$$g_1(X^1) < g_1(X^2) < g_1(X^2) < \ldots ,$$

(10)

$$g_2(X^1) > g_2(X^2) > g_2(X^2) > \ldots$$

The solution X^i is an adjacent basic solution to X^{i-1} and X^{i+1}. Adjacent basic solutions differ in one basic variable only. From a basic e.s., adjacent basic e.s. can be determinated investigating the reduced cost matrix CR. A column vector CR(i, j) of dimension 2 in CR, associated with a nonbasic arc (i, j) is efficient, if CR(i, j) = 0 and there exists a vecor of weights $\lambda_i = (\lambda_1, \lambda_2)$ such that

$$\lambda \mathrm{CR} \geq 0 \text{ and } \lambda \mathrm{CR}(i, j) = 0.$$

It is known that any basis of BF(X) may be represented as a rooted spanning tree with a root in the node s and n–1 basic arcs. Let $u^1(i)$ and $u^2(i)$ be dual variables (potentials) associated with a node i for the first and the second objective functions respectively. The potentials of the node j, which is the ending node of the arc (i, j) in the spanning tree T, are determined by the equations

$$u^1(j) = u^1(i) + a^1_{ij};$$
$$u^2(j) = u^2(i) + a^1_{ij}.$$

For each arc $(i, j) \in U$, the components of the vector CR(i, j) are determined as follows:

$$\mathrm{CR}^1(i, j) = u^1(i) - u^1(j) + a^1_{ij};$$
$$\mathrm{CR}^2(i, j) = u^2(i) - u^2(j) + a^2_{ij}.$$

Moving from X^1 towrads X^{i+1} for obtaining the basic tree associated with X^{i+1}, we must remove an arc from the tree T^1, which corresponds to X^1 and pivot another arc. The appropriate arc, which is going to enter the basis, is that arc, which obtains a minimum increase in the first objective for a unit decrease in the second objective. To satisfy (10), the potentials of this arc must satisfy the inequalities:

$$\mathrm{CR}^1(i, j) > 0,$$
$$\mathrm{CR}^2(i, j) < 0.$$

We determine the function $d(i, j)$ on the set of nonbasic arcs as follows:

$$d(i, j) = \begin{cases} |\mathrm{CR}^1(i, j)/\mathrm{CR}^2(i, j)| \text{ if } \mathrm{CR}^1(i, j)>0 \text{ and } \mathrm{CR}^2(i, j)<0, \\ \alpha \text{ otherwise.} \end{cases}$$

The arc (p, l) which has to be pivoted into the basis has to have the value of $d(p, l)$ which is:

$$d(p, l) = \min \{ d(i, j) / (i, j) \in U \text{ and is a nonbasic arc} \}.$$

From the equation in [5] it follows that the components of the corresponding arc vector λ are determined as follows:

(11)
$$\lambda_1 = d(p, l) / (1 + d(p, l)),$$
$$\lambda_2 = 1 / (1 + d(p, l)).$$

In order to initiate step 1 of the algorithm we need to see that the initial basic e.s. X may be determined solving a single criteria min-cost flow problem using a basic method [7]. The objective function of this problem is a weighted objective function $(1-\theta)\, g_1(Y) + g_2(Y)$, where $0 < \theta < 0{,}1$. To obtain successive basic e.s. we determine the arc (p, l) and its potential $d(p, l)$. The new weight vector λ can be determined from (11).

There are two ways of finding the next solution. The one is the customary way – denote by λ the new weighted objective function and solve the min-cost problem. The other one uses the spanning tree corresponding to the previous solution.

The removing of the arc (q, l) of the tree T^i breaks into two disjoint subtrees T_1^i and T_2^i, where the first of them contains the node q and the second one – the node l. Then we can change the value of the potential ui just of the nodes in the subtree T_2^i, adding to them the value $CR^i(p, l)$.

3.2. Finding the e.s. of the problem $BF_x(Y)$

To find e.s. of the problem $BF_x(Y)$ in step 3 we can use a modification of the negative cycles method proposed by H u [8], which changes the flow over the cycles with a negative or zero value. The "costs" of the arcs are the corresponding coeffcicents in the weighted objective function. The obtained solution may not be a basic solution.

4. Some other properties of the MFP

Having in mind that the solution of BFP is considerably easier, the theorem given below enables the recursive finding of the e.s. of MFP for $k>2$.

Let $MF_{k-1}(X)$ be a MFP with $k-1$ criteria $g_1, g_2, \ldots, g_{k-1}$. The numbers $\lambda_i \geq 0$, $i \in I_{k-1}$, $\sum_{i \in I_{k-1}} \lambda_i = 1$ are given. We define the function

(12)
$$F_{k-1}(X) = \sum_{i \in I_{k-1}} \lambda_i g_i(X).$$

Theorem 3. The e.s. of the BFP with objectives F_{k-1} and $g_k(x)$ is an e.s. of $MF(X)$ and the reverse.

Proof. If X is an e.s. of BFP, then two numbers μ_1 and μ_2 exist, which make the function

$$F_k(X) = \mu_1 F_{k-1}(X) + \mu_2 g_k(X)$$

take its minimal value for X and from (12) it follows:

$$F_k = \mu_1 \lambda_1 g_1 + \ldots + \mu_1 \lambda_{k-1} g_{k-1} + \mu_2 g_k.$$

This is a weighted objective function for MFP, because

$$\mu_1\lambda_1 + \ldots + \mu_1\lambda_{k-1} + \mu_2 = \mu_1(\lambda_1 + \ldots + \lambda_{k-1}) + \mu_2 = 1,$$

i.e. X is an e.s. for this problem.

The reverse follows from the equalities

$$F_k = \lambda_1 g_1 + \ldots + \lambda_{k-1} g_{k-1} + \lambda_k g_k = (1-\lambda_k)(\lambda_1 g_1/(1-\lambda_k) + \ldots + \lambda_{k-1} g_{k-1}/(1-\lambda_k)) + \lambda_k g_k.$$

Conclusion

In this paper we have presented some properies of the efficient solution of the MFP. We have developed a method for solving large scale bicriteria network flow problem. The method determines all non-dominated flows from a single source node to the single sink node in the network. The method uses the property which states that each non-dominated solution of the investigated problem may be represented as a sum of a basic feasible solution of an appropriatly defined network problem for bicriteria circulation. Using this property we may preserve the "destruction" of costraint matrix.

In the worst case the number of e.s. increases exponentially with the size of the problem. Theoretically, for $k>2$, we can use the property stated in Theorem 3. The problem for finding e.s. of MFP, which satisfy given conditions with the help of specialized flow algorithms, is still unsolved.

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Свойства эффективных решений мультикритериальной проблеммы потока в сети

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(Р е з ю м е)

Рассматривается задача о потоке в сети, когда каждой дуге в сети сопоставлены несколько параметров. Дефинированы соответствующие линейные функции дуг сети для разных параметров и эти функции минимизируются. Поставленная задача является многокритериальной задачей для потока в сети. Обсуждаются проблеммы нахождения эффективных решений (э.р.) задачи, сохраняя унимодулярность матрицы ограничений. Доказаны свойства э.р. Предложен метод нахождения э.р. двукритериальной задачи, определяя каждое из них как сумма базисного решения задачи о минимальном потоке и эффективное решение дефинированной потоковой задачи.